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Discrete Approximation of ECG Signals

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Abstract

In this research we investigate the non-equidistant discretization generated by Blaschke functions and its application in the compression of ECG signals. This discretization provides interpolation at chosen points of the signal, gives more detail near important regions and cares less about constant-like parts. We focus mainly on the explanation of how to generate this kind of discretization. Questions arise from finding suitable parameters for the system applied as well as from generating the set of discrete points. We suggest possible solutions for these problems and show an example of application on a real signal.

Keywords: discrete approximation, non-equidistant discretization, Blaschke functions, ECG signal

MSC: 30J10, 68W25

1. Introduction

In medical sciences and especially in diagnostics electrocardiograms (also known as ECG signals), which describe the functioning of the human heart, are of great importance. Their mathematical modelling, approximation, denoising, segmenting and analysis are growing fields of research. Convenient storage of huge amounts of ECG data requires efficient and robust compression methods and algorithms.

Besides the standard methods of spectral analysis and wavelet techniques (see topical review [1]) the application of rational complex functions, such as Blaschke functions seems very promising. These functions and related systems have already been successfully applied to solve various problems e.g. in control theory.

In this paper we focus mainly on the explanation and illustration of how to generate a non-equidistant discretization (or non-uniform division) using the above mentioned functions. We also give an example of application on a real signal.

By examining the values of the signal in these discrete points, we acquire good compression. Signals such as ECG curves have intervals with rapid change and great variance in value, and have intervals with slow, calm change. This discretization gives more detail near important regions and cares less about constant-like parts. This property makes it more suitable for ECG signals than a uniform disretization with the same number of points. Furthermore, it provides interpolation at discrete points of the signal, and one point can be arbitrarily chosen.

2. Discretization

In this section we introduce the Blaschke functions, recall and illustrate some of their important properties, the definition of the argument function, and analyse the inverse function. We will show how to obtain a non-equidistant discretization.

2.1. Blaschke functions

Let us denote by $\mathbb{C}$ the set of complex numbers and let $\mathbb{D} := \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk and $\mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \}$ the unit circle.

Definition 2.1. Given $a \in \mathbb{D}$ and $d \in \mathbb{T}$ we define the *Blaschke function* as

$$B_{a,d} \colon \mathbb{C} \to \mathbb{C} \qquad B_{a,d}(z) := d \cdot \frac{z-a}{1-\overline{a}z}.$$

Note that a is the zero of the function $B_{a,d}$ and $B_{a,d}$ has a pole of order one at $1/\overline{a} \in \mathbb{C} \setminus (\mathbb{D} \cup \mathbb{T})$.

We state an important property of these functions in the following theorem.

Theorem 2.2. *The Blaschke function $B_{a,d}$ ($a \in \mathbb{D},\ d \in \mathbb{T}$) maps $\mathbb{T}$ on $\mathbb{T}$. (It is a bijection of $\mathbb{T}$, a so-called 'onefold' map.)*

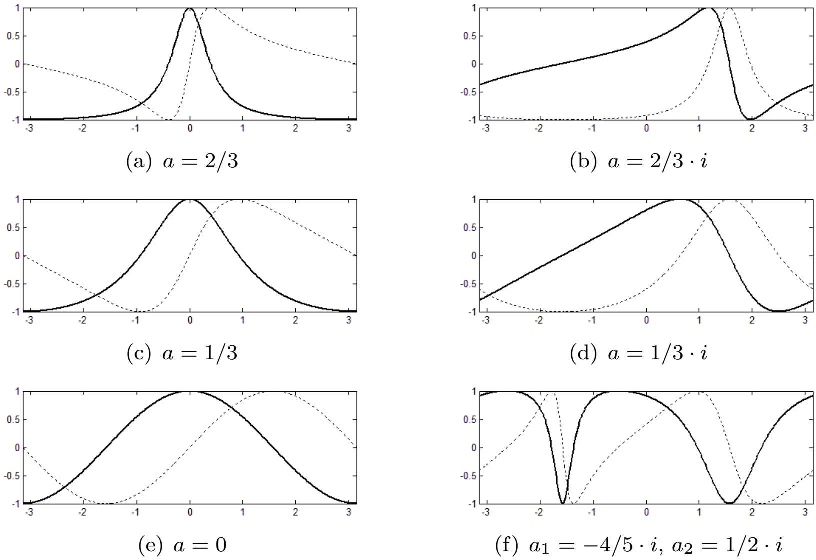
Proof. This theorem comes from a few short calculations. First one may calculate that $|B_{a,d}(z)| = 1$ with $a \in \mathbb{D}, d \in \mathbb{T}$ and $z \in \mathbb{T}$. Then forming the inverse function of $B_{a,d}$ (which turns out to be the Blaschke function $B_{-ad,\overline{d}}$) concludes the proof. □

Remark 2.3. A Blaschke function is also a $\mathbb{D} \to \mathbb{D}$ bijection.

Definition 2.4. A *Blaschke product* is the product of Blaschke functions. I.e. given $1 < m \in \mathbb{N}$, $a_1, \ldots, a_m \in \mathbb{D}$ and $d_1, \ldots, d_m \in \mathbb{T}$ a Blaschke product of order m is defined as follows:

$$B_{(a_1,d_1),\ldots,(a_m,d_m)}(z) := \prod_{j=1}^{m} B_{a_j,d_j}(z).$$

Remark 2.5. A Blaschke product of order m is an m-fold map on $\mathbb{T}$, i.e. for all $w \in \mathbb{T}$ there exists exactly m values, $z_1, \ldots, z_m \in \mathbb{T}$, that are mapped to w. (This becomes easy to see after defining the argument function in Subsection 2.2.)



(a) $a = 2/3$ (b) $a = 2/3 \cdot i$

(c) $a = 1/3$ (d) $a = 1/3 \cdot i$

(e) $a = 0$ (f) $a_1 = -4/5 \cdot i$, $a_2 = 1/2 \cdot i$

Figure 1: Blaschke functions. Values on $\mathbb{T}$ are shown.

It is difficult to visualize complex functions. Nevertheless we are now only interested in the function values on $\mathbb{T}$. One can plot these by mapping a segment of the real line on $\mathbb{T}$ and observing the real and imaginary parts of the function values. We shall use the one-to-one map

$$[-\pi, \pi) \ni t \mapsto e^{it} \in \mathbb{T}$$

with the complex exponential function.

Example 2.6. Figure 1 shows 5 examples of Blaschke functions varying the parameter a (the parameter $d = 1$ is fixed) and 1 example of a Blaschke product ($d_1 = d_2 = 1$). The values on $\mathbb{T}$ are shown, the real part is drawn with a solid line, the imaginary part with a dashed line.

Note that $a = 0$ gives the special case when $B_{0,1}(z) = z$, thus we obtain the well known sine and cosine functions.

While $|a|$ is growing (a is getting closer to $\mathbb{T}$) one can observe a more and more fast change in the function values at (and near) the argument of a, and a slower change on the opposite side. The change in the argument (the angle) of a is reflected by the horizontal positions of rapid and slow changes.

In the case of Blaschke products we may observe an even greater variety in the speed function values change.

2.2. The argument function

The fact that $B_{a,d}$ is a $\mathbb{T} \to \mathbb{T}$ bijection enables us to analyse the relation between the angle (the argument) of $z \in \mathbb{T}$ and its image $B_{a,d}(z)$. Recall that in this case $z = e^{it}$ for some $t \in [-\pi, \pi)$.

Definition 2.7. Define the *argument function* associated to the Blaschke function $B_{a,d}$ ($a \in \mathbb{D}, d \in \mathbb{T}$) as follows:

$$\beta_{a,d} \colon [-\pi, \pi) \to \mathbb{R} \qquad \beta_{a,d}(t) := \arg B_{a,d}(e^{it}).$$

We can extend $\beta_{a,d}$ to $\mathbb{R}$. In order to make $\beta_{a,d}$ continuous on $\mathbb{R}$, define

$$\beta_{a,d} \colon \mathbb{R} \to \mathbb{R} \quad \text{so that} \quad \beta_{a,d}(t + 2\pi) = \beta_{a,d}(t) + 2\pi$$

is satisfied.

Theorem 2.8. *The explicit form of $\beta_{a,d}$ with $a = r \cdot e^{i\varphi}$ ($r \in [0, 1) \subset \mathbb{R}, \varphi \in \mathbb{R}$) and $d = e^{i\delta}$ ($\delta \in [-\pi, \pi)$) is*

$$\beta_{a,d}(t) = (\delta + \varphi) + 2 \arctan \left(\frac{1 + r}{1 - r} \tan \frac{t - \varphi}{2} \right).$$

Furthermore $\beta_{a,d}$ is strictly increasing and invertible.

Proof. The deduction of the previous formula and other necessary calculations can be found e.g. in [2], see also [5]. The strictly increasing property follows from the fact that the first derivative $\beta'_{a,d}$ is positive. Finally the inverse function can be calculated with little effort. □

Remark 2.9. Note that $d \in \mathbb{T}$ is of little importance in this context. The parameter $d = e^{i\delta}$ can be chosen so that $\beta_{a,d}$ maps $[-\pi, \pi)$ on $[-\pi, \pi)$. That is why d is sometimes omitted and β_a is written. In this case we consider d to be the one with the above mentioned handy property. (The Blaschke functions—a subset of those—are also defined sometimes using only one parameter, namely a.)

Definition 2.10. The argument function of a Blaschke product is defined as

$$\beta_{a_1,\ldots,a_m}(t) := \frac{1}{m} \arg \prod_{j=1}^{m} B_{a_j(,d_j)}(z) = \frac{1}{m} \sum_{j=1}^{m} \beta_{a_j}(t).$$

This definition makes use of the fact that the argument of the product of two complex numbers is the sum of the arguments of each. Also the $1/m$ factor is applied to maintain the $[-\pi, \pi)$ bijection property.

Remark 2.11. The function $\beta_{a_1,\ldots,a_m}$ is also strictly increasing but its inverse function has no known explicit form.

Example 2.12. Figure 2 shows 5 examples of argument functions of a Blaschke function and 1 example of an argument function associated to a Blaschke product. These functions correspond to the Blaschke functions (and product) on Figure 1. Note the relation between the speed of change in function value in the case of Blaschke functions and the slope of the corresponding argument function.

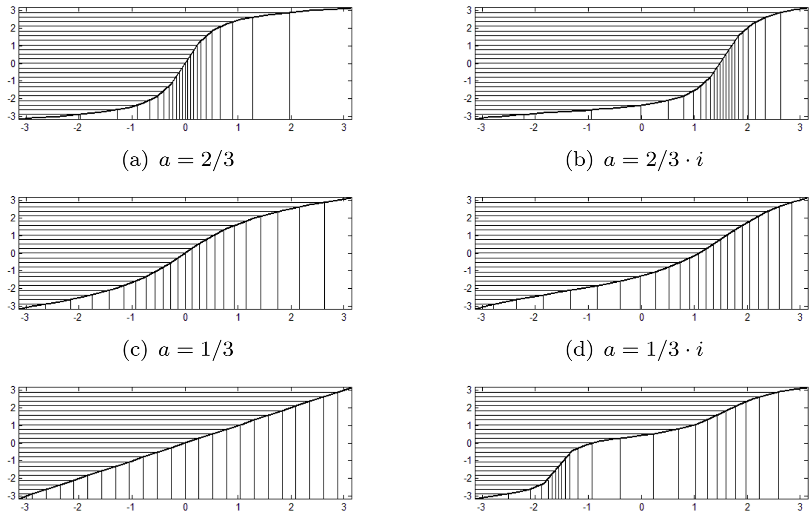


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Figure 2: Argument functions and NEDs.

2.3. Generating a non-equidistant discretization

Now everything is ready to define a non-equidistant discretization (NED) on the interval $[-\pi, \pi)$. The argument function defined in Subsection 2.2 shall be used. The inverse image of an equidistant discretization will be considered.

Definition 2.13. Consider the set of $N \in \mathbb{N}$ equidistant points

$$D_0^N := \left\{ -\pi + k \cdot \frac{2\pi}{N} \ : \ 0 \leq k \leq N-1 \right\} \subset [-\pi, \pi).$$

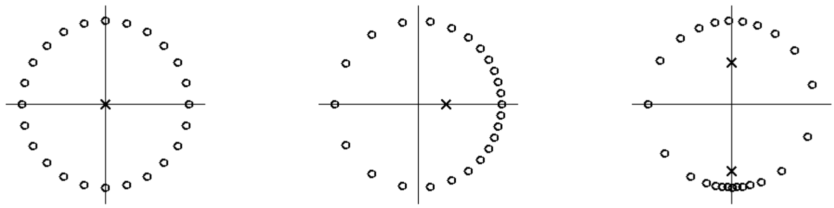
Then for given $1 \leq m \in \mathbb{N}$, and $a_1, \ldots, a_m \in \mathbb{D}$ parameters the set

$$D_{a_1,\ldots,a_m}^N := \left\{ \beta_{a_1,\ldots,a_m}^{-1}(t) \ : \ t \in D_0^N \right\} \subset [-\pi, \pi)$$

is a *non-equidistant discretization (NED)* of N points on the interval $[-\pi, \pi)$.

Example 2.14. Figure 2 also shows examples of NEDs defined by the functions in question with $N = 24$ and with various values of a. Figure 3 shows 3 of the previous NEDs by plotting the corresponding points on the complex unit circle. (Actually 3(a) shows an ED.)

Observe that the points of the discretization are dense at (and near) the argument(s) of the parameter(s) a, and sparse on the 'opposite side'. Moreover, the closer a is to $\mathbb{T}$, the bigger is the change in density of the points. So by choosing a (or more parameters) well, we can get the desired property: more points at one region, less points at other intervals.



(a) $a = 0$ (b) $a = 1/3$ (c) $a_1 = -4/5 \cdot i$, $a_2 = 1/2 \cdot i$

Figure 3: NEDs on the unit circle $\mathbb{T}$.

Remark 2.15. We anticipated that one point of the discretization $D^N_{a_1,\ldots,a_n}$ can be an arbitrarily fixed $p_0 \in [-\pi, \pi)$ point. By choosing $\nu \in [0, \frac{2\pi}{N})$ and forming the $D^{N,\nu}_{a_1,\ldots,a_n}$ inverse image of the set $D^N_0 + \nu$ we can achieve that $p_0 \in D^{N,\nu}_{a_1,\ldots,a_n}$. Let $\nu = \beta_{a_1,\ldots,a_n}(p_0) - k_0 \cdot \frac{2\pi}{N}$ with $k_0 \in \mathbb{N}$ so that $\nu \in [0, \frac{2\pi}{N})$.

2.4. Calculating the inverse

In this subsection we examine the generation of the NED, the calculation of the inverse images.

The inverse of an argument function β_a associated to a Blaschke function has an explicit form as stated in Theorem 2.8. So in this case we can easily calculate the inverse images of given points.

But in general, the inverse of an argument function $\beta_{a_1,\ldots,a_m}$ associated to a Blaschke product has no known explicit form as mentioned in Remark 2.11. So in this case we have to solve N non-linear equations. This can be done in several ways: applying the bisection method, or applying Newton's method which converges faster (in order 2) but has to be given a close enough starting point. As usual, the combination of these methods can be fruitful. One may apply the bisection method for the points in a clever order.

3. Application on ECG signals

In this section we outline a possible method as the application of the discretization defined above in Section 2. We discuss an example in the case of ECG signals. A signal is also to be considered as a function on the $[-\pi, \pi)$ interval.

We can outline the application of a suitable NED for a signal as follows:

- First we must find suitable parameters for our signal, i.e. define $m, a_1, \ldots, a_m$ and ν. This could be done by e.g. using Monte Carlo methods or a simplex algorithm (see [4]) etc.

- Generate NED according to the previously defined parameters (and corresponding β argument functions) as described in detail in Section 2.
- Examine and store values at the discrete points of the signal defined by points of the NED. (Interpolation may be needed if a point falls between sampling points of the signal.) Note that the points of the NED need not to be stored but only the parameters defining them—which also saves space.

In the case of ECG signals, the point which can be arbitrarily chosen should be the location of the peak of the R (usually the highest) wave since this point is important in diagnostics.

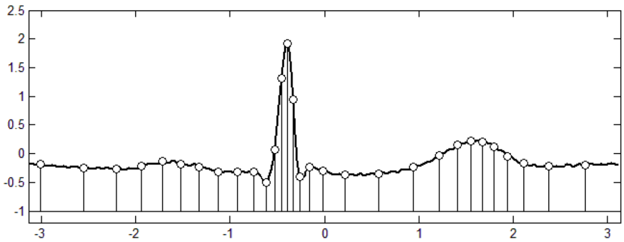


Figure 4: Application: an example. (Signal: mitdb/103/1/10139–10462)

Example 3.1. Figure 4 shows a sample application. The ECG segment from the database [3] is interpolated at points of a NED with manually chosen parameters

$$N = 30,\ m = 3,\ a_1 = \frac{4}{5} \cdot e^{-\frac{1}{8}\cdot\pi},\ a_2 = \frac{13}{20} \cdot e^{\frac{13}{24}\cdot\pi},\ a_3 = \frac{9}{20} \cdot e^{-\frac{1}{2}\cdot\pi},\ \nu = \frac{\pi}{60}.$$

Notice that important points of the signal are preserved during the discretization and observe the adaptive change in density of NED points.

4. Future work

Related to the research presented in this paper we may mention these areas of further investigation:

- Wider application, detailed measurement and validation for ECG signals and maybe other similar signals.
- Further measurements on the required calculations generating a NED.
- Elaboration and examination of methods for finding suitable parameters for a signal, including a hyperbolic approach which the unit disk and the Blaschke functions are closely related to.
- Construction of efficient algorithms for compression and reconstruction of signals.

5. Conclusions

In this paper we suggested a possible method for processing and compressing e.g. ECG signals. The main idea and subject of explanation was the generation of a non-equidistant discretization of a real interval using rational complex functions, namely the Blaschke functions and their associated argument functions.

This approach provides a very elegant way to handle more or less detail needed at different regions of a signal.

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