Proceedings of the 8th International Conference on Applied Informatics
Eger, Hungary, January 27–30, 2010. Vol. 1. pp. 53–60.

Partial Approximative Set Theory: A View from Galois Connections

Zoltán Csajbók

Department of Health Informatics
University of Debrecen Faculty of Health
e-mail: csajzo@de-efk.hu

Abstract

The rough set theory was introduced by the Polish mathematician, Z. Pawlak in the early 1980s. It was a new mathematical approach to *vagueness*. His idea can be formulated in two different ways, namely, in a point-wise and in a point-free manner. Its points-wise generalization can be described, e.g., on the language of relation algebra, and it is a well-known theory.

In this paper, our starting point will be an *arbitrary* family of subsets of the universe. Neither that it covers the universe nor that the universe is finite will be assumed. Moreover, within this framework, our concepts of lower and upper approximations are straightforward *point-free* generalizations of Pawlak's ones. We shall investigate by what conditions such so-called weak lower and upper approximations form Galois connections. We will compare our result to the similar result given by the point-wise generalization.

Keywords: Vagueness, approximation of sets, rough set theory, partial approximative set theory, Galois connection.

MSC: Primary 06A15; Secondary 03B52.68T37.

1. Introduction

The rough set theory was introduced by the Polish mathematician, Z. Pawlak in the early 1980s [6], [7]. It was a new mathematical approach to vagueness [8].

Let U be a *finite* set of distinguishable objects, called the universe of discourse, and $\varepsilon \subseteq U \times U$ be an equivalence relation on U. The elements of partition generated by ε are called ε-elementary sets. An arbitrary subset $X \subseteq U$ can be naturally approximated by two sets, on one hand, by the union of all the ε-elementary sets that are subsets of X, called the lower ε-approximation of X, on the other hand by the union of all the ε-elementary sets that have a non-empty intersection with X, called the upper ε-approximation of X.

The basic idea of the Pawlak's rough set theory is that the vagueness of a set is described by the difference of its ε-upper and ε-lower approximations which is called the ε-boundary of the set. A set is *rough* if its ε-boundary is non-empty.

In this paper, our starting point will be an *arbitrary* family of subsets of U. Neither that this family of sets covers the universe nor that the universe is finite will be assumed. In order that the vagueness can be treated in a general approximate framework, let our initial concept be the following: a pair of maps $f, g : 2^U \to 2^U$ is a *weak approximation pair* on U if $\forall X \in 2^U \ (f(X) \subseteq g(X))$. As Düntsch and Gediga noticed in [3], this constraint seems to be the weakest condition for a sensible concept of approximations of subsets in U. Moreover, the maps f, g is a *strong approximation pair* on U if each subset $X \in 2^U$ is bounded by $f(X)$ and $g(X)$, i.e., $\forall X \in 2^U \ (f(X) \subseteq X \subseteq g(X))$ [3]. Approved the hypothesis that the notion of "approximation" may be mathematically modelled by the notion of Galois connections [5], we also have to study under what conditions a weak and/or strong approximation pair forms a Galois connection? In this paper we will treat only the question of weak approximations.

The paper is organized as follows. In Section 2 we summarize some basic notations. Section 3 presents some fundamental concepts and their properties of the classical Pawlak's rough set theory. In short, we also show its point-wise generalization. In Section 4 we outline the basic concepts of the partial approximative set theory which is a straightforward point-free generalization of Pawlak's theory. In this new framework, we define weak lower and upper approximate maps. We investigate what conditions they become weak and strong approximation pairs and under what conditions the weak approximation pair forms a Galois connection.

2. Basic Notions and Notations

Let U be any set. Let $\mathfrak{A} \subseteq 2^U$ be a family of sets of which elements are subsets of U. The union of $\mathfrak{A}$ is $\bigcup \mathfrak{A} = \{x \mid \exists A \in \mathfrak{A}(x \in A)\}$, and the intersection of $\mathfrak{A}$ is $\bigcap \mathfrak{A} = \{x \mid \forall A \in \mathfrak{A}(x \in A)\}$. If $\mathfrak{A}$ is an empty family of sets we define $\bigcup \emptyset = \emptyset$ and $\bigcap \emptyset = U$.

If ϵ is an arbitrary binary relation on U, let $[x]_\epsilon$ denote the ϵ-related elements to x, i.e., $[x]_\epsilon = \{y \in U \mid (x, y) \in \epsilon\}$.

A non-empty set P together with a partial order $\leq$ on P is called a *poset*, in symbols, $P = (P, \leq)$.

Let $(P, \leq_P)$ and $(Q, \leq_Q)$ be two posets. A map $f : P \to Q$ is *monotone* if $x \leq_P y \Rightarrow f(x) \leq_Q f(y)$, *antitone* if $x \leq_P y \Rightarrow f(y) \leq_Q f(x)$.

A pair (f, g) of maps $f : P \to Q$, $g : Q \to P$ is a *(monotone) Galois connection* or an *adjunction* between P and Q if

$$\forall p \in P \, \forall q \in Q \ (f(p) \leq_Q q \Leftrightarrow p \leq_P g(q)).$$

f is called the *lower adjoint* and g the *upper adjoint* of the Galois connection.

We also write a quadruple (P, f, g, Q) for a whole Galois connection. If $P = Q$ it is said (P, f, g, P) is a Galois connection on P.

The following theorem gives a useful characterization of Galois connections.

Theorem 2.1 ([4], Lemma 79). *(P, f, g, Q) is a Galois connection if and only if*

1. *$p \leq_P g(f(p))$ for all $p \in P$ and $f(g(q)) \leq_Q q$ for all $q \in Q$;*
2. *the maps f and g are monotone.*

3. Fundamentals of Rough Set Theory

The basic concepts and properties of rough set theory can be found, e.g, in [7], [4]. Here we cite only a few of them which will be important in what follows. We partly restate these well-known facts on the language of approximations.

Definition 3.1. A pair (U, ε), where U is a finite universe of discourse and ε is an equivalence relation on U, is called a *Pawlak's approximation space.*

A subset $X \subseteq U$ is *ε-definable*, if it is a union of ε-elementary sets, otherwise X is *ε-undefinable.* By definition, the empty set is considered to be an ε-definable set. Let $\mathfrak{D}_{U/\varepsilon}$ denote the family of ε-definable subsets of U.

In Pawlak's approximation spaces, the lower and upper approximations of X can be defined in two equivalent forms, namely, in a *point-free* manner—based on the ε-elementary sets, and in a *point-wise* manner—based on the elements.

Definition 3.2. Let a Pawlak's approximation space (U, ε) and a subset $X \in 2^U$ be given. The *lower ε-approximation* of X is

$$\underline{\varepsilon}(X) \quad = \quad \bigcup\{Y \mid Y \in U/\varepsilon, Y \subseteq X\} = \{x \in U \mid [x]_\varepsilon \subseteq X\}, \tag{3.1}$$

and the *upper ε-approximation* of X is

$$\overline{\varepsilon}(X) \quad = \quad \bigcup\{Y \mid Y \in U/\varepsilon, Y \cap X \neq \emptyset\} = \{x \in U \mid [x]_\varepsilon \cap X \neq \emptyset\}. \tag{3.2}$$

The set $B_\varepsilon(X) = \overline{\varepsilon}(X) \setminus \underline{\varepsilon}(X)$ is the *ε-boundary* of X.

X is *ε-crisp*, if $B_\varepsilon(X) = \emptyset$, otherwise X is *ε-rough.*

It follows just from the definitions that $\underline{\varepsilon}(X), \overline{\varepsilon}(X) \in \mathfrak{D}_{U/\varepsilon}$, the maps $\underline{\varepsilon}, \overline{\varepsilon} : 2^U \to \mathfrak{D}_{U/\varepsilon}$ are total, and, in general, many-to-one.

Based on binary relations on U, lower and upper ε-approximations can be generalized via their *point-wise* definitions [4].

Definition 3.3. Let ϵ be an arbitrary binary relation on U and $X \in 2^U$.

The *lower ϵ-approximation* of X is

$$\underline{\epsilon}(X) \quad = \quad \{x \in U \mid [x]_\epsilon \subseteq X\}, \tag{3.3}$$

and the *upper ϵ-approximation* of X is

$$\overline{\epsilon}(X) \quad = \quad \{x \in U \mid [x]_\epsilon \cap X \neq \emptyset\}. \tag{3.4}$$

If ϵ^{-1} denotes the inverse relation of ϵ, in the same manner one can also define lower and upper ϵ^{-1}-approximations.

Theorem 3.4 ([4], Proposition 134). *Let ϵ be an arbitrary binary relation on U. Then $(2^U, \overline{\epsilon}, \underline{\epsilon^{-1}}, 2^U)$ and $(2^U, \overline{\epsilon^{-1}}, \underline{\epsilon}, 2^U)$ are Galois connections on $(2^U, \subseteq)$.*

Some other properties of lower and upper ϵ-approximations are expressed by some properties of binary relations, and vice versa.

Theorem 3.5. *Let ϵ be an arbitrary binary relation on U.*

1. *The pair of maps $\underline{\epsilon}, \overline{\epsilon}$ is a weak approximation pair if and only if ϵ is connected.*
2. *The pair of maps $\underline{\epsilon}, \overline{\epsilon}$ is a strong approximation pair if and only if ϵ is reflexive.*
3. *$(2^U, \overline{\epsilon}, \underline{\epsilon}, 2^U)$ is a Galois connection on $(2^U, \subseteq)$ if and only if ϵ is symmetric.*

Proof. In [4]. *1.* Proposition 136., *2.* Proposition 137., *3.* Proposition 138. □

The next example shows that if the relation ϵ is symmetric, it is not sufficient that the lower and upper ϵ-approximations defined in a *point-free* manner form a Galois connection.

Example 3.6. Let $U = \{x_1, x_2, x_3\}$ be the universe.
$\epsilon = \{(x_1, x_1), (x_1, x_2), (x_2, x_1), (x_2, x_3), (x_3, x_2)\} \subset U \times U.$
$[x_1]_\epsilon = \{u \in U \mid (x_1, u) \in \epsilon\} = \{x_1, x_2\},$
$[x_2]_\epsilon = \{u \in U \mid (x_2, u) \in \epsilon\} = \{x_1, x_3\},$
$[x_3]_\epsilon = \{u \in U \mid (x_3, u) \in \epsilon\} = \{x_2\}.$
$U/\epsilon = \{[x_1]_\epsilon, [x_2]_\epsilon, [x_3]_\epsilon\} = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2\}\}.$

For example, the point-free definitions of lower and upper ϵ-approximations of $\{x_2\}$ are the following:
$\underline{\epsilon}_{pf}(\{x_2\}) = \bigcup\{Y \mid Y \in U/\epsilon, Y \subseteq \{x_2\}\} = \bigcup\{[x_3]_\epsilon\} = \bigcup\{\{x_2\}\} = \{x_2\},$
$\overline{\epsilon}_{pf}(\{x_2\}) = \bigcup\{Y \mid Y \in U/\epsilon, Y \cap \{x_2\} \neq \emptyset\} = \bigcup\{[x_1]_\epsilon, [x_3]_\epsilon\}$
$\qquad = \bigcup\{\{x_1, x_2\}, \{x_2\}\} = \{x_1, x_2\}$

Do $\{x_2\} \subseteq \underline{\epsilon}_{pf}(\overline{\epsilon}_{pf}(\{x_2\}))$ and/or $\overline{\epsilon}_{pf}(\underline{\epsilon}_{pf}(\{x_2\})) \subseteq \{x_2\}$ hold?
$\underline{\epsilon}_{pf}(\overline{\epsilon}_{pf}(\{x_2\})) = \underline{\epsilon}_{pf}(\{x_1, x_2\}) = \bigcup\{[x_1]_\epsilon\} = \bigcup\{\{x_1, x_2\}\} = \{x_1, x_2\} \supseteq \{x_2\}$
$\overline{\epsilon}_{pf}(\underline{\epsilon}_{pf}(\{x_2\})) = \overline{\epsilon}_{pf}(\{x_2\}) = \{x_1, x_2\} \nsubseteq \{x_2\}$

That is, by Theorem 2.1, $(2^U, \underline{\epsilon}_{pf}, \overline{\epsilon}_{pf}, 2^U)$ does not form a Galois connection.

Of course, the point-wise definitions of lower and upper ϵ-approximations of $\{x_2\}$, by *3.* of Theorem 3.5, fulfil the two formulas in question, i.e., if
$\underline{\epsilon}_{pw}(\{x_2\}) = \{x \in U \mid [x]_\epsilon \subseteq \{x_2\}\} = \{x_3\},$
$\overline{\epsilon}_{pw}(\{x_2\}) = \{x \in U \mid [x]_\epsilon \cap \{x_2\} \neq \emptyset\} = \{x_1, x_3\}$, then
$\underline{\epsilon}_{pw}(\overline{\epsilon}_{pw}(\{x_2\})) = \underline{\epsilon}_{pw}(\{x_1, x_3\}) = \{x \in U \mid [x]_\epsilon \subseteq \{x_1, x_3\}\} = \{x_2\} \supseteq \{x_2\},$
$\overline{\epsilon}_{pw}(\underline{\epsilon}_{pw}(\{x_2\})) = \overline{\epsilon}_{pw}(\{x_3\}) = \{x \in U \mid [x]_\epsilon \cap \{x_3\} \neq \emptyset\} = \{x_2\} \subseteq \{x_2\}.$

4. Partial Approximative Set Theory

In practice there are properties which do not characterize all members of an observed collection of objects. A very simple example, when one investigates an infinite set via a finite family of its finite subsets. For instance, a number theorist studies regularities of natural numbers using computers.

Throughout this section let U be any non-empty set.

4.1. Base Systems

Definition 4.1. Let $\mathfrak{B} \subseteq 2^U$ be a non-empty family of non-empty subsets of U called the *base system*. Its elements are the $\mathfrak{B}$-*sets*.

A family of sets $\mathfrak{D} \subseteq 2^U$ is $\mathfrak{B}$-*definable* if its elements are $\mathfrak{B}$-sets, otherwise $\mathfrak{D}$ is $\mathfrak{B}$-*undefinable*. A non-empty subset $X \in 2^U$ is $\mathfrak{B}$-*definable* if there exists a $\mathfrak{B}$-definable family of sets $\mathfrak{D}$ such that $X = \bigcup \mathfrak{D}$, otherwise X is $\mathfrak{B}$-*undefinable*. The empty set is considered to be a $\mathfrak{B}$-definable set.

Let $\mathfrak{D}_{\mathfrak{B}}$ denote the family of $\mathfrak{B}$-definable sets of U.

Definition 4.2. The base system $\mathfrak{B} \subseteq 2^U$ is *single-layered*, if

$$\forall B \in \mathfrak{B}\ \forall \mathfrak{B}' \subseteq \mathfrak{B} \setminus \{B\}\, (B \cap \bigcup \mathfrak{B}' \neq B).$$

Assumed that the intended meaning of a base system $\mathfrak{B}$ is a collection of primitive properties, then, in words, a base system $\mathfrak{B}$ is single-layered if every $\mathfrak{B}$-definable subset of the universe has at least one element which can be characterized by exactly one primitive property. Or else in a more general context, $\mathfrak{B}$ is single-layered if every subset of the universe has at least one element which can be characterized by at most one primitive property. The most simple way to construct a single-layered base system from any base system $\mathfrak{B}$ is to form an intersection structure from $\mathfrak{B}$. If $\mathfrak{X}$ is a family of subsets of U, it is an *intersection structure* if $\forall \mathfrak{X}'(\neq \emptyset) \subseteq \mathfrak{X}\ (\bigcap \mathfrak{X}' \in \mathfrak{X})$ [2].

Lemma 4.3. *The base system* $\mathfrak{B}$ *is single-layered and* $\forall D \in \mathfrak{D}_{\mathfrak{B}}\, \forall B \in \mathfrak{B}\ (B \cap D = B \Leftrightarrow B \cap D \neq \emptyset)$ *holds if and only if the* $\mathfrak{B}$*-sets are pairwise disjoint.*

Proof. ($\Rightarrow$) By a contradiction, let us assume that the $\mathfrak{B}$-sets are not pairwise disjoint, that is, $\exists B_1, B_2 \in \mathfrak{B}(B_1 \neq B_2 \wedge B_1 \cap B_2 \neq \emptyset)$. Because the base system $\mathfrak{B}$ is single-layered, neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$ hold. Consequently, $\exists B_2 \in \mathfrak{D}_{\mathfrak{B}}\, \exists B_1 \in \mathfrak{B}\ (B_1 \cap B_2 \neq \emptyset \wedge B_1 \cap B_2 \neq B_1)$, a contradiction.

($\Leftarrow$) Since, the $\mathfrak{B}$-sets are pairwise disjoint, the base system $\mathfrak{B}$ is trivially single-layered.

When $D = \emptyset$, $B \cap D = B \Leftrightarrow B \cap D \neq \emptyset$ trivially holds. If $D \neq \emptyset$, of course, $B \cap D = B \Rightarrow B \cap D \neq \emptyset$ clearly satisfies as well. So, it is enough to prove that $\forall D(\neq \emptyset) \in \mathfrak{D}_{\mathfrak{B}}\, \forall B \in \mathfrak{B}\ (B \cap D \neq \emptyset \Rightarrow B \cap D = B)$.

Let us take an arbitrary $\mathfrak{B}$-definable set $D = \bigcup \mathfrak{B}'(\neq \emptyset) \in \mathfrak{D}_{\mathfrak{B}}$ ($\mathfrak{B}' \subseteq \mathfrak{B}$), and an arbitrary $\mathfrak{B}$-set $B \in \mathfrak{B}$. By the distributive law,

$$B \cap D \neq \emptyset \Leftrightarrow B \cap \bigcup \mathfrak{B}' \neq \emptyset \Leftrightarrow \bigcup_{B' \in \mathfrak{B}} (B \cap B') \neq \emptyset.$$

Since $\mathfrak{B}'$ consists of pairwise disjoint $\mathfrak{B}$-sets, there exists a $B' \in \mathfrak{B}'$ such that $B = B'$. Consequently, $B \cap D = B \cap \bigcup \mathfrak{B}' = B$. □

4.2. Weak Lower and Upper Approximations

Definition 4.4. Let $\mathfrak{B} \subseteq 2^U$ be a base system and X be any subset of U.

The *weak lower* $\mathfrak{B}$*-approximation* of X is

$$\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \subseteq X\}, \tag{4.1}$$

and the *weak upper* $\mathfrak{B}$*-approximation* of X is

$$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \bigcup \{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\}. \tag{4.2}$$

Notice that $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ and $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ are straightforward *point-free* generalizations of lower and upper ε-approximations. Clearly, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X), \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) \in \mathfrak{D}_{\mathfrak{B}}$, and the maps $\mathfrak{C}^{\flat}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$, $\mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to \mathfrak{D}_{\mathfrak{B}}$ are total, onto, and, in general, many-to-one. Furthermore, both of them are monotone.

Theorem 4.5. *Let the fixed base system $\mathfrak{B} \subseteq 2^U$ and maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ and $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ be given.*

1. $\forall X \in 2^U(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$.
2. $\forall X \in 2^U(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) \subseteq X)$*—that is, $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is contractive.*
3. $\forall X \in 2^U(X \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$ *if and only if* $\bigcup \mathfrak{B} = U$*—that is, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is extensive if and only if $\mathfrak{B}$ covers the universe.*

Proof. *1.* and *2.* are straightforward.

3. ($\Rightarrow$) $U \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(U) = \bigcup\{A \mid A \in \mathfrak{B}, A \subseteq U\} = \bigcup \mathfrak{B}$, $\bigcup \mathfrak{B} \subseteq U$, so $\bigcup \mathfrak{B} = U$.

($\Leftarrow$) $\forall X \in 2^U(X \subseteq U = \bigcup \mathfrak{B})$, and we have $X \subseteq \bigcup(\mathfrak{B} \setminus \{A \mid A \in \mathfrak{B}, A \cap X = \emptyset\}) = \bigcup\{A \mid A \in \mathfrak{B}, A \cap X \neq \emptyset\} = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)$. □

In other words, the pair of maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}} : 2^U \to 2^U$ is a weak approximation pair on U, and it is a strong one if and only if the base system $\mathfrak{B}$ covers the universe.

Remark 4.6. If a non-empty $X \in 2^U$ does not have any non-empty $\mathfrak{B}$-definable subsets, then $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \emptyset = \emptyset \subseteq X$—which, however, does not provide new information about the relationship between X and $\mathfrak{B}$. On the other hand, if $\bigcup \mathfrak{B} \neq U$, then $\forall X \subseteq U \setminus \bigcup \mathfrak{B}\ \forall B \in \mathfrak{B}(X \cap B = \emptyset)$. Consequently, for all these subsets $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \bigcup \emptyset = \emptyset$, i.e., the empty set is the weak upper $\mathfrak{B}$-approximation of certain non-empty subsets of U. All these uncommon cases may be interpreted so that our knowledge about the universe encoded in the base system is incomplete. These uncommon cases may be excluded by two partial maps called strong lower and strong upper $\mathfrak{B}$-approximations. For more details, see [1].

Definition 4.7. Let the fixed base system $\mathfrak{B} \subseteq 2^U$ and maps $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ and $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ be given. The quadruple $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$ is called a *weak* $\mathfrak{B}$*-approximation space.*

4.3. Galois Connections on Weak $\mathfrak{B}$-approximation Spaces

Given a weak $\mathfrak{B}$-approximation space $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$, let us investigate what conditions have to be satisfied by $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$ in order that the quadruple $(2^U, \mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, 2^U)$ forms a Galois connection.

Clearly, the maps $\mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}} : 2^U \to 2^U$ are monotone, and *2.* of Theorem 2.1 trivially holds.

The next theorem shows that the first half of *1.* of Theorem 2.1 satisfies if and only if the base system covers the universe.

Theorem 4.8. *Let the weak $\mathfrak{B}$-approximation space $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$ be given. Then $\forall X \in 2^U\ (X \subseteq \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)))$ if and only if $\bigcup \mathfrak{B} = U$.*

Proof. $(\Rightarrow)$ By a contradiction, let us assume that $\bigcup \mathfrak{B} \neq U$. Accordingly, $\exists X'(\neq \emptyset) \subseteq U \setminus \bigcup \mathfrak{B}$. Hence, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X')) = \emptyset$, which gives $\emptyset \neq X' \subseteq \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X')) = \emptyset$, a contradiction.

$(\Leftarrow)$ Let $Y' \in \{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\}$. If so, $Y' \subseteq \bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap X \neq \emptyset\} = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)$. Therefore, $Y' \in \{Y \mid Y \in \mathfrak{B}, Y \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)\}$, i.e., $Y' \subseteq \bigcup\{Y \mid Y \in \mathfrak{B}, Y \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)\} = \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$. Hence, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) \subseteq \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$. Since $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is contractive, then $\mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)) \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X)$ also holds. Consequently, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$.

At last, since $\bigcup \mathfrak{B} = U$, then $\mathfrak{C}^{\sharp}_{\mathfrak{B}}$ is extensive, thus $X \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(X) = \mathfrak{C}^{\flat}_{\mathfrak{B}}(\mathfrak{C}^{\sharp}_{\mathfrak{B}}(X))$. □

Taking up the question of the second half of *1.* of Theorem 2.1, in general, the condition $\forall X \in 2^U\ (\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X)) \subseteq X)$ also does not hold.

Theorem 4.9. *Let the weak $\mathfrak{B}$-approximation space $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$ be given. Then the base system $\mathfrak{B}$ is single-layered and $\forall X \in 2^U\ (\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X)) \subseteq X)$ if and only if the $\mathfrak{B}$-sets are pairwise disjoint.*

Proof. $(\Rightarrow)$ By a contradiction, let us assume that the $\mathfrak{B}$-sets are not pairwise disjoint, that is, $\exists B_1, B_2 \in \mathfrak{B}\ (B_1 \neq B_2 \wedge B_1 \cap B_2 \neq \emptyset)$. Because the base system $\mathfrak{B}$ is single-layered, neither $B_1 \subseteq B_2$ nor $B_2 \subseteq B_1$ hold. Hence, e.g., for B_1, we have

$$\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1)) = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(B_1) = \bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap B_1 \neq \emptyset\},$$

thus $B_1 \cup B_2 \subseteq \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1))$, and so $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(B_1)) \nsubseteq B_1$, a contradiction.

$(\Leftarrow)$ Obviously, if the $\mathfrak{B}$-sets are disjoint, the base system $\mathfrak{B}$ is single-layered. Furthermore, $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(\emptyset)) = \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\emptyset) = \emptyset \subseteq \emptyset$ holds, independently of that the $\mathfrak{B}$-sets are pairwise disjoint or not.

Let $\emptyset \neq X \in 2^U$. If $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \emptyset$, then $\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\emptyset) = \emptyset \subseteq X$. Let $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \mathfrak{B}' \neq \emptyset$ for a family of $\mathfrak{B}$-sets $\mathfrak{B}' \subseteq \mathfrak{B}$. Of course, $\mathfrak{C}^{\flat}_{\mathfrak{B}}(X) = \bigcup \mathfrak{B}' \subseteq X$ because of the map $\mathfrak{C}^{\flat}_{\mathfrak{B}}$ is contractive. By Lemma 4.3, we get

$$\begin{aligned}\mathfrak{C}^{\sharp}_{\mathfrak{B}}(\mathfrak{C}^{\flat}_{\mathfrak{B}}(X)) &= \mathfrak{C}^{\sharp}_{\mathfrak{B}}(\bigcup \mathfrak{B}') = \bigcup\{Y \mid Y \in \mathfrak{B}, Y \cap \bigcup \mathfrak{B}' \neq \emptyset\} \\ &= \bigcup\{Y \mid Y \in \mathfrak{B}, Y \subseteq \bigcup \mathfrak{B}'\} \subseteq \bigcup\{Y \mid Y \in \mathfrak{B}, Y \subseteq X\} \subseteq X.\end{aligned}$$

□

The results of Theorem 4.8 and 4.9 can be summarized in the following corollary.

Corollary 4.10. *Let the weak $\mathfrak{B}$-approximation space $(U, \mathfrak{B}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, \mathfrak{C}^{\sharp}_{\mathfrak{B}})$ be given.*

Then $(2^U, \mathfrak{C}^{\sharp}_{\mathfrak{B}}, \mathfrak{C}^{\flat}_{\mathfrak{B}}, 2^U)$ forms a Galois connection if and only if the base system $\mathfrak{B}$ is a partition of the universe U.

Acknowledgements. I would like to thank Tamás Mihálydeák for valuable discussions and suggestions.

References

[1] Csajbók, Z., Partial Approximative Set Theory, In *Programs, Proofs, Processes, Sixth Conference on Computability in Europe*, CiE 2010, Ponta Delgada (Azores), Portugal, June 30 - July 4, 2010, Abstract and Handout Booklet. To appear.

[2] Davey, B.A. – Priestley, H.A., *Introduction to Lattices and Order*, Second edition, Cambridge University Press, Cambridge (2002)

[3] Düntsch, I. – Gediga, G., *Approximation Operators in Qualitative Data Analysis*, In: H. de Swart et al. (Eds.): TARSKI, Lecture Notes in Computer Science Vol. 2929, Springer-Verlag, Berlin, Heidelberg, 214–230 (2003).

[4] Järvinen, J., Lattice theory for rough sets. In: *Transactions on Rough Sets VI.* LNCS, vol. 4374, Springer, Heidelberg, 400–498 (2007).

[5] Pagliani, P. – Chakraborty, M., *A Geometry of Approximation. Rough Set Theory: Logic, Algebra and Topology of Conceptual Patterns*, Trends in Logic, Vol. 27, Springer, 2008.

[6] Pawlak, Z., Rough Sets, *International Journal of Information and Computer Science*, Vol. 11(5), 341–356 (1982)

[7] Pawlak, Z., *Rough Sets: Theoretical Aspects of Reasoning about Data*, Kluwer Academic Publishers, Dordrecht (1991)

[8] Read, S., *Thinking about Logic: An Introduction to the Philosophy of Logic*, Oxford University Press, Oxford (1995).

Zoltán Csajbók
Department of Health Informatics
University of Debrecen Faculty of Health
Sóstói út 2-4, H-4400 Nyíregyháza, Hungary