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Bivariate Difference-differential Dimension Polynomials and Their Computation in Maple*

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Abstract

In this paper we present Maple implementations of two algorithms developed by M. Zhou and F. Winkler for computing a relative Gröbner basis of a finitely generated difference-differential module and for computing the bivariate difference-differential dimension polyomial of the module with respect to the natural bifiltration of the ring of difference-differential operators.

The notion of relative Gröbner basis and its use for computing bivariate difference-differential dimension polynomials is explained. After this the implementations of the two algorithms are illustrated by a couple of examples.

Keywords: difference-differential dimension polynomials, relative Gröbner bases

MSC: 12H99

1. Introduction

The role of Hilbert polynomials in commutative algebra, algebraic geometry and combinatorics is well known [5, 1]. The theory of Gröbner bases provides an efficient method to compute Hilbert polynomials of filtered (and thus also graded) modules over polynomial rings [2]. The equivalent to the Hilbert polynomial in differential algebra is the differential dimension polynomial which was introduced by Kolchin [7]. For a system of algebraic differential equations the corresponding differential dimension polynomial describes the number of arbitrary constants in the general solution of the system. The analytic interpretation of Kolchin's differential dimension polynomial is connected to the theory of relativity (especially gravitation theory). Einstein [4] describes the strength of a system of partial differential equations governing a physical field by a certain function associated to the system. It is

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known [14] that this function coincides with the appropriate differential dimension polynomial. In differential algebra the theory of Gröbner bases in modules over rings of differential operators was developed by Mikhalev and Pankratev [14, 15], Oaku and Shimoyama [16], Insa and Pauer [6] and gives methods for computation of differential dimension polynomials, see [8].

The equivalent to the Hilbert polynomial in difference-differential algebra is the difference-differential dimension polynomial which was introduced by Levin [9, 10]. Together with Mikahlev [12, 13] he advanced this concept to difference-differential modules and field extensions, see also [8]. Important approaches to using Gröbner bases methods for the computation of difference-differential dimension polynomials are due to Levin [11] as well as Zhou and Winkler [17, 18, 19, 20]. In the sense of Einstein computing the difference-differential dimension polynomial is equivalent to determining the strength of a system of differential equations with delay governing a physical field.

Problem 1.1. Suppose we are given two difference-differential equations governing a physical field

$$\frac{\partial}{\partial t} h_1(t, x+1) + h_2(t, x-2) = 0, \quad \text{and} \quad \frac{\partial^2}{\partial t^2} h_1(t, x+1) + \frac{\partial}{\partial t} h_2(t, x) = 0. \tag{1.1}$$

What is the relation between the number of arbitrary constants in the general solution (h_1, h_2) of this system and degree bounds for h_1 and h_2 in t and x?

2. Preliminaries

In this paper $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{Q}$ will denote the sets of all integers, all nonnegative integers and all rational numbers, respectively. We assume all rings to have a unit element, every subring of a ring contains the ring's unit element. Ring homomorphisms are considered to be unitary, i.e., mapping unit element to unit element. By the module over a ring R we always mean a unitary left R-module.

Definition 2.1. 1. Let R be a commutative ring,

$$\{d_1, \ldots, d_m\} \qquad \text{and} \qquad \{s_1, \ldots, s_n\}$$

sets of mutually commuting derivations and automorphisms on R, respectively, i.e., $u \circ v = v \circ u$ for all $u, v \in \{d_1, \ldots, d_m\} \cup \{s_1, \ldots, s_n\}$. Then R is called a *difference-differential ring* with basic set of derivations $\{d_1, \ldots, d_m\}$ and basic set of automorphisms $\{s_1, \ldots, s_n\}$. If R is also a field then it is called a *difference-differential field.*

2. Let $\Delta = \{\delta_1, \ldots, \delta_m\}$ and $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$. By Λ we denote the commutative semigroup

$$\{\delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \mid k_1, \ldots, k_m \in \mathbb{N},\ l_1, \ldots, l_n \in \mathbb{Z}\}.$$

Elements of Λ are called *difference-differential terms.*

3. The free R-module generated by Λ we will denote by D. Hence elements of D are of the form $\sum_{\lambda\in\Lambda} a_\lambda \lambda$ with $a_\lambda \in R$ and only finitely many a_λ are not vanishing. D can be equipped with a natural ring structure with the commutation rules $\alpha\beta = \beta\alpha$, $\delta_i a = a\delta_i + d_i(a)$, $\sigma_j a = s_j(a)\sigma_j$ for all $a \in R$, $\alpha, \beta \in \Delta \cup \Sigma$, $1 \le i \le m$, $1 \le j \le n$. The obtained ring is called the *ring of difference-differential operators* over R.

4. The order of $\lambda = \delta_1^{k_1} \cdots \delta_m^{k_m} \sigma_1^{l_1} \cdots \sigma_n^{l_n} \in \Lambda$ is given by $\operatorname{ord}\lambda = k_1 + \ldots + k_m + |l_1| + \ldots + |l_n|$ and the order of $\vartheta = \sum_{\lambda\in\Lambda} a_\lambda \lambda \in D$ is given by

$$\operatorname{ord}\vartheta = \max\{\operatorname{ord}\lambda \mid a_\lambda \neq 0\}.$$

The order of λ in $\delta_1, \ldots, \delta_m$ and $\sigma_1, \ldots, \sigma_n$ is given by $\operatorname{ord}_\delta \lambda = k_1 + \ldots + k_m$ and $\operatorname{ord}_\sigma \lambda = |l_1| + \ldots + |l_n|$, respectively. The order of θ in $\delta_1, \ldots, \delta_m$ and $\sigma_1, \ldots, \sigma_n$ is given by

$$\operatorname{ord}_\delta \vartheta = \max\{\operatorname{ord}_\delta \lambda \mid a_\lambda \neq 0\}, \quad \text{and} \quad \operatorname{ord}_\sigma \vartheta = \max\{\operatorname{ord}_\sigma \lambda \mid a_\lambda \neq 0\},$$

respectively.

5. A left D-module is called *difference-differential module.*

6. For all $r, s \in \mathbb{Z}$ let $D_{rs} = \{\theta \in D \mid \operatorname{ord}_\delta \theta \le r,\ \operatorname{ord}_\sigma \theta \le s\}$ if $r, s \in \mathbb{N}$ and $D_{rs} = \{0\}$ if at least one of the numbers r, s is negative. Then $(D_{rs})_{r,s\in\mathbb{Z}}$ is called *natural bifiltration* of D.

7. Let M be a difference-differential module with generators $h_1, \ldots, h_q$ and for $r, s \in \mathbb{Z}$ let $M_{rs} := D_{rs}h_1 + \ldots + D_{rs}h_q$. Then the family $(M_{rs})_{r,s\in\mathbb{Z}}$ is called *excellent bifiltration* of M.

3. Relative Gröbner bases and bivariate difference-differential dimension polynomials

3.1. Bivariate difference-differential dimension polynomials

When computing a Hilbert polynomial of a polynomial module one usually considers the growth of order in all variables. In a difference-differential setting such an approach obviously does not take care of the nature of difference-differential modules. It is much more convenient to consider the growth of order of the derivations and automorphisms separately (see [11]).

Theorem 3.1. *Let K be a difference-differential field, $\Delta = \{\delta_1, \ldots, \delta_m\}$, $\Sigma = \{\sigma_1, \ldots, \sigma_n\}$, D the ring of difference-differential operators over K with natural bifiltration $(D_{rs})_{r,s\in\mathbb{Z}}$, M a finitely generated difference-differential module, and $(M_{rs})_{r,s\in\mathbb{Z}}$ an excellent bifiltration of M. Then there exists a polynomial $\psi(r,s) \in \mathbb{Q}[r,s]$ such that*

1. *$\psi(r,s) = \dim_K M_{rs}$ for all sufficiently large $r, s \in \mathbb{Z}$.*

2. $\deg_r \psi(r,s) \le m$, $\deg_s \psi(r,s) \le n$ *and for all* $i = 0,\dots,m$, $j = 0,\dots,n$ *there exist* $a_{ij} \in \mathbb{Z}$ *such that*

$$\psi(r,s) = \sum_{i=0}^{m}\sum_{j=0}^{n} a_{ij}\binom{r+i}{i}\binom{s+j}{j}.$$

Definition 3.2. The polynomial ψ whose existence is stated by Theorem 3.1 is called the *bivariate difference-differential dimension polynomial* of M associated with the excellent bifiltration $(M_{rs})_{r,s\in\mathbb{Z}}$.

3.2. Relative Gröbner bases

Computation of Gröbner bases in rings of difference-differential operators needs to take care of problems which arise from noncommutativity. Since we want to deduce a bivariate difference-differential dimension polynomial we also have to choose a special reduction to reflect the decomposition of $\Delta\cup\Sigma$ into Δ and Σ. For a detailed description of term orders in a ring of difference-differential operators we refer to [19, 20].

Definition 3.3. Let $E \ni e, \tilde{e}$ be a finite, totally ordered set and for $k, r \in \mathbb{N}^m$, $l, s \in \mathbb{Z}^n$ let $\lambda = \delta^k\sigma^l$ and $\mu = \delta^r\sigma^s$. We define two total orders $\prec$ and $\prec'$ on ΛE by

$$\begin{aligned}\lambda e \prec \mu\tilde{e} \quad :\Longleftrightarrow \quad & (\mathrm{ord}_\sigma(\lambda), \mathrm{ord}_\delta(\lambda), e, k_1,\dots,k_m, |l_1|,\dots,|l_n|, l_1,\dots,l_n)\\ & <_{\mathrm{lex}} (\mathrm{ord}_\sigma(\mu), \mathrm{ord}_\delta(\mu), \tilde{e}, r_1,\dots,r_m, |s_1|,\dots,|s_n|, s_1,\dots,s_n),\end{aligned}$$

and

$$\begin{aligned}\lambda e \prec' \mu\tilde{e} \quad :\Longleftrightarrow \quad & (\mathrm{ord}_\delta(\lambda), \mathrm{ord}_\sigma(\lambda), e, k_1,\dots,k_m, |l_1|,\dots,|l_n|, l_1,\dots,l_n)\\ & <_{\mathrm{lex}} (\mathrm{ord}_\delta(\mu), \mathrm{ord}_\sigma(\mu), \tilde{e}, r_1,\dots,r_m, |s_1|,\dots,|s_n|, s_1,\dots,s_n).\end{aligned}$$

M. Zhou and F. Winkler also extended the definition of reduction and of Göbner bases appropriately.

Definition 3.4. Let F be a finitely generated free difference-differential module and $f, g \in F\setminus\{0\}$. If there exists a $\lambda\in\Lambda$ such that

$$\mathrm{lt}_\prec(\lambda g) = \mathrm{lt}_\prec(f) \qquad \text{and} \qquad \mathrm{lt}_{\prec'}(\lambda g) \preceq' \mathrm{lt}_{\prec'}(f)$$

then we say that f *can be* $\prec$*-reduced to* $f-\lambda g$ *relative to* $\prec'$. Let W be a submodule of F, and let $G = \{g_1,\dots,g_p\}$ be a subset of $W\setminus\{0\}$. Then G is called a $\prec$*-Gröbner basis of* W *relative to* $\prec'$ if and only if every $f \in W\setminus\{0\}$ can be $\prec$-reduced to 0 modulo G relative to $\prec'$.

Remark 3.5. Of course the notions of relative reduction and relative Gröbner basis are also applicable to other term orders on ΛE. We refer to [19, 20].

Zhou and Winkler [19] proved the following Theorem which is the key to computing bivariate difference-differential dimension polynomials of finitely generated difference-differential modules.

Theorem 3.6. *Let K be a difference-differential field and M a finitely generated left D-module with set of generators $\{h_1, \ldots, h_q\}$. Let F be a free difference-differential module with basis $\{e_1, \ldots, e_q\}$ and $\pi : F \to M$ the natural difference-differential epimorphism of F onto M, i.e., π is given by $\pi(e_i) = h_i$ for $i = 1, \ldots, q$. Let N be the submodule of F given by $N = \ker \pi$ and let $G = \{g_1, \ldots, g_p\}$ be a $\prec$-Gröbner basis of N relative to $\prec'$. For $r, s \in \mathbb{N}$ let*

$$\begin{aligned} U_{rs} \;\; = \;\; & \{\omega e \in \Lambda E \mid \operatorname{ord}_\Delta(\omega) \le r,\; \operatorname{ord}_\sigma(\omega) \le s, \\ & \text{and } \operatorname{ord}_\delta(\operatorname{lt}_{\prec'}(\lambda g)) > r \text{ for all } \lambda \in \Lambda,\; g \in G \text{ s.t. } \omega e = \operatorname{lt}_\prec(\lambda g)\}. \end{aligned}$$

Then for any $r, s \in \mathbb{N}$ the set $\pi(U_{rs})$ is a basis of the vector K-space M_{rs}. In particular for sufficiently large r, s the bivariate difference-differential dimension polynomial ψ associated with M satisfies $\psi(r, s) = |U_{rs}|$.

4. Implementation examplified

The two algorithms for computing relative Gröbner bases and bivariate difference-differential dimension polynomials are provided as a Maple package. The purpose of the package is to provide methods for computation of bivariate difference-differential dimension polynomials where the difference-differential field K is the quotient field of the ring of polynomials with rational coefficients in a user-defined number of variables.

The package is loaded as follows.

```
> libname := libname,"/path/to/DiffDiff.mla":
> with(DifferenceDifferential);
```

[DDDPol]

From now on we will assume that the package is always loaded.

The package only contains the procedure `DDDPol` which takes as input a description of the ring D of difference-differential operators and as output provides procedures for computing relative Gröbner bases and bivariate difference-differential dimension polynomials. For a detailed description of the input possibilities we refer to [3].

Example 4.1. Let $K = \mathbb{Q}(t, x)$ and consider the system (1.1) of difference-differential equations. For $f(t, x) \in K$ define a derivative d and an automorphism s on K by $d(f(t, x)) := \frac{\partial}{\partial t} f(t, x)$ and $s(f(t, x)) := f(t, x + 1)$. Let $\Delta = \{\delta\}$ and $\Sigma = \{\sigma\}$, where δ and σ are associated with d and s, respectively. Then (1.1) determines a difference-differential module $M = Dh_1 + Dh_2$, where h_1, h_2 satisfy the defining equations (see also Example 4.3 of [19])

$$\begin{aligned} \delta\sigma h_1 + \sigma^{-2} h_2 &= 0, \\ \delta^2 \sigma h_1 + \delta h_2 &= 0. \end{aligned}$$

Then M is isomorphic to the factor-module of a free difference-differential module F with free generators $e_1,\ e_2$ by the difference-differential submodule N generated by

$$G = \{g_1 = \delta\sigma e_1 + \sigma^{-2} e_2,\ g_2 = \delta^2 \sigma e_1 + \delta e_2\}.$$

Note that in the notation of Theorem 3.6 G is a basis for $\ker(\pi)$. First we have to call `DDDPol` according to $\Delta = \{\delta\}$, $\Sigma = \{\sigma\}$ and $E = \{e_1,\ e_2\}$, declaring derivatives and automorphisms appropriately (note that we also have to specify the inverse of the automorphism s).

```
> DD1 := DDDPol("variables" = [t,x], "derivatives" = [proc (f) diff(f, t)
end proc], "automorphisms" = [[proc (f) subs(x = x+1, f) end proc, proc (f)
subs(x = x-1, f) end proc]], "generators" = [e[1], e[2]]);
```

DD1 := table(["DimPol" = DimPol, "relGB" = relGB])

We assign the output of `DDDPol` to `DD1`. Then we call `relGB` from the `table` generated in the first step. with the list `[delta*sigma*e[1]+e[2]/sigma^2 , delta^2*sigma*e[1]+delta*e[2]]` as argument representing the set G.

```
> DD1["relGB"]([delta*sigma*e[1]+e[2]/sigma^2,delta^2*sigma*e[1]+delta*e[2]]);
```

[delta*sigma*e[1]+e[2]/sigma^2 , delta^2*sigma*e[1]+delta*e[2] ,
delta*e[2]/sigma-delta*e[2]/sigma^3 , delta^2*e[1]/sigma+delta*e[2]]

The output means that

$$\begin{aligned} G' \quad = \quad & \{g_1 = \delta\sigma e_1 + \sigma^{-2} e_2,\ g_2 = \delta^2\sigma e_1 + \delta e_2, \\ & g_3 = \delta\sigma^{-1} e_2 - \delta\sigma^{-3} e_2,\ g_4 = \delta^2 \sigma^{-1} e_1 + \delta e_2\} \end{aligned}$$

is a $\prec$-Gröbner basis of the free difference-differential submodule N relative to $\prec'$. This coincides with [19] (actually g_3 coincides up to the factor $-\sigma$ with the corresponding element of the relative Gröbner basis obtained by Zhou and Winkler but the two Gröbner bases practically are the same, i.e., they generate the same submodule). The bivariate difference-differential dimension polynomial is computed by the procedure `DimPol`:

```
> DD1["DimPol"]([delta*sigma*e[1]+e[2]/sigma^2, delta^2*sigma*e[1]+delta*e[2],
delta*e[2]/sigma-delta*e[2]/sigma^3, delta^2*e[1]/sigma+delta*e[2]]);
```

2*r+5+4*s

I.e., there are $2r+5+4s$ arbitrary constants in the general solution of the system (1.1) if the degree bound for t, resp. x, is determined by r, resp. s, accordingly.

Example 4.2. Let $\Delta = \{\delta_1, \delta_2\}$, $\Sigma = \{\sigma\}$. Let $M = Dh$, where h satisfies the defining equations (see also Example 4.2 in [19])

$$(\delta_1^4\delta_2\sigma^{-3} + \delta_1^2\delta_2\sigma^3)h = 0 \quad \text{and} \quad (\delta_1^2\delta_2\sigma^2 - \delta_1^2\delta_2\sigma^{-4})h = 0.$$

Then M is isomorphic to the factor module of a free difference-differential module F with free generator e by the difference-differential submodule N generated by

$$G = \{g_1 = \delta_1^4\delta_2\sigma^{-3}e + \delta_1^2\delta_2\sigma^3 e,\ g_2 = \delta_1^2\delta_2\sigma^2 e - \delta_1^2\delta_2\sigma^{-4} e\}.$$

First we call `DDDPol` with the appropriate arguments (since all the coefficients are equal to 1, the result is independent of the actual choice of derivatives and automorphism).

```
> DD2 := DDDPol("noder" = 2, "noaut" = 1, "generators" = [e]);
```

DD2 := table(["DimPol" = DimPol, "relGB" = relGB])

Then the bivariate difference-differential dimension polynomial is computed as follows:

```
> DD2["DimPol"]([delta[1]^4*delta[2]*e/sigma^3+delta[1]^2*
delta[2]*sigma^3*e, delta[1]^2*delta[2]*sigma^2*e-delta[1]^2*
delta[2]*e/sigma^4]);
```

6*s*r+15*r-30

which coincides with the result of Zhou and Winkler.

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