Proceedings of the 8th International Conference on Applied Informatics
Eger, Hungary, January 27–30, 2010. Vol. 1. pp. 203–210.

Considerations on Offsetting Plane Curves

Gábor Valasek, János Vida

Eötvös Loránd University, Faculty of Informatics

e-mail: `{valasek,jvida}@inf.elte.hu`

Abstract

Offsetting operations in geometric modeling are inevitable for CAD/CAM, and graphical applications. Offsets of integral parametric polynomial curves are usually not polynomial, the offsetting operation is closed only for a subset of rational parametric curves.

The theoretical results of Pottmann et al. clarified the problem, the solution is using rational curves with rational offsets. However, this conversion of modeling software implies certain implementation costs, and might increase interrogation time.

That is why further attempts for finding less complicated, and/or less time consuming solutions, with similar practical functionality, might still be of interest.

Our paper briefly surveys the most relevant achievements in the field, tries to suggest possible simpler alternatives, based on the interpolation of point, direction, and curvature data, at the ends of curve segments, then examines, and evaluates them in detail. Our purpose is to make it easy to understand and measure real functional and numerical limitations.

Keywords: Geometric Modeling, CAGD, CAD/CAM, Computer Graphics

MSC: 65D17, 65D18, 68U07

1. Exact representation of offset curves

Let us first briefly review the results of Farouki and Pottmann on subsets of integral and polynomial curves that have rational offsets.

Let us consider regular parametric curves, $\mathbf{r}(\xi)$. We can express their derivative in the form

$$\mathbf{r}'(\xi) = \mathbf{t}(\xi)\sigma(\xi),$$

where $\mathbf{t}(\xi) = \frac{\mathbf{r}'(\xi)}{|\mathbf{r}'(\xi)|}$ denotes the unit tangent vector, and $\sigma(\xi)$ is the parametric speed: $\sigma(\xi) = |\mathbf{r}'(\xi)| = \sqrt{x'^2(\xi) + y'^2(\xi)}$.

The offset at distance d is defined as

$$\mathbf{r}_d(\xi) = \mathbf{r}(t) + d\mathbf{n}(\xi),$$

where $\mathbf{n}(\xi) = \frac{(y'(\xi), -x'(\xi))}{\sqrt{x'^2(\xi)+y'^2(\xi)}}$ is the unit normal vector.

Due to the square root in the normal vector, the offset curve is algebraically more complex than its progenitor, which, in practice, means that the offset of an integral polynomial curve is neither an integral, nor a rational polynomial curve in general.

To overcome this problem, Farouki and Sakkalis proposed the use of Pythagorean Hodograph (PH) curves [2], polynomials that have hodographs satisfying the following Pythagorean condition:

$$x'^2(t) + y'^2(t) = \sigma^2(t), \tag{1.1}$$

for some integral polynomial $\sigma(t)$. These curves have *rational offsets* of degree $2n-1$, and in addition, they also provide *exact rectification*, since the arc-length function

$$s(\xi) = \int_0^{\xi} \sigma(t)dt$$

is a polynomial. This makes PH curves remarkably useful in certain applications, see for example [3].

The constraint (1.1) on the hodograph decreases the flexibility of the PH curves: a PH curve of degree $n \geq 3$ has only $n+3$ scalar degrees of freedom, while a general integral polynomial plane curve of the same degree has $2(n+1)$ scalar degrees of freedom. In practice, further special properties have to be taken into account as well, for example the lowest degree PH curve that can have an inflection is the quintic PH curve.

Pottmann [5] investigated the set of rational curves with rational offsets (RCWRO). We can express a curve as the envelope of its $\mathbf{g}(t)$ tangent line family:

$$\mathbf{g}(t) : n_x(t)x + n_y(t)y = h(t), \tag{1.2}$$

where $h(t)$ is the signed distance of the tangent line $\mathbf{g}(t)$ from the origin, and $\mathbf{n}(t)$ is the rational unit normal vector of the tangent line $\mathbf{g}(t)$. If $h(t)$ is rational and $\mathbf{n}(t)$ is of the form

$$n_x(t) = \frac{2a(t)b(t)}{a^2(t)+b^2(t)}, \qquad n_y(t) = \frac{a(t)^2 - b(t)^2}{a^2(t)+b^2(t)},$$

for some polynomials $a(t), b(t)$, then the envelopes of such tangent lines will be rational curves with rational offset. Moreover, the offset curve's degree is the same as of the original curve, since only $h(t)$ has to be replaced by $h_d(t) = h(t) + d$ in (1.2).

Once again, due to the construction, certain limitations have to be taken into account. E. g. the lowest degree rational curves with rational offsets that offer the same flexibility as conics, are RCWRO quartics. The arc-length function can no longer be expressed by elementary functions. Pottmann gave a formulation for the subset of the RCWRO curves that have rational arc-length functions, see [5] for more details.

2. Point-normal-curvature interpolation

2.1. The interpolation problem

Let there be given endpoint position, tangent direction and curvature data

$$(\mathbf{p}_i \in \mathbb{E}^2, \mathbf{t}_i \in \mathbb{R}^2, \kappa_i \in \mathbb{R}), \quad (i = 0, 1)$$

Find a polynomial curve in Bézier form

$$\mathbf{r}(t) = \sum_{i=0}^{n} \mathbf{b}_i B_i^n(t)$$

that reconstructs the above quantities at its endpoints, that is for $i = 0, 1$

$$\mathbf{r}(i) = \mathbf{p}_i, \quad \mathbf{t}(i) = \mathbf{t}_i, \quad \kappa(i) = \kappa_i, \tag{2.1}$$

where $\mathbf{t}(t)$ and $\kappa(t)$ denote the tangent and curvature functions of the curve $\mathbf{r}(t)$.

The interpolation condition poses 4-4 scalar constraints on the solution curve at its endpoints.

2.2. Indirect solution to the interpolation problem

Let us find a solution for the above problem in two steps:

- construct two local Bézier curve solutions, $\mathbf{c}_0(t)$ and $\mathbf{c}_1(t)$, such that at $t = i$ the curve $\mathbf{c}_i$ interpolates the given point, tangent and curvature $(\mathbf{p}_i, \mathbf{t}_i, \kappa_i)$ data $(i = 0, 1)$
- smooth the two local solution curves into a single global Bézier curve, so that the position, tangent and curvature data at the endpoints remain unchanged

2.2.1. Integral quadratic local solution

Let us find a quadratic Bézier solution for the local interpolation problems. We only consider interpolating $(\mathbf{p}_0, \mathbf{t}_0, \kappa_0)$ at $t = 0$, the case of $(\mathbf{p}_1, \mathbf{t}_1, \kappa_1)$ can be derived similarly. Let us only consider the case $\kappa_0 \neq 0$, since if $\kappa_0 = 0$, the curve becomes a line through $\mathbf{p}_0$ with tangent vector $\mathbf{t}_0$.

The conditions of (2.1) result in the following constraints on the control points $\mathbf{b}_0, \mathbf{b}_1, \mathbf{b}_2$ of $\mathbf{r}(t) = \mathbf{c}_0(t)$:

$$\mathbf{b}_0 = \mathbf{p}_0$$

$$\mathbf{b}_1 = \mathbf{p}_0 + \lambda \mathbf{t}_0$$

for some $\lambda > 0$ displacement along the tangent line. Using the curvature equation for quadratic Bézier curves at $t = 0$

$$\kappa_0 = \frac{1}{2} \frac{\Delta \mathbf{b}_0 \times \Delta^2 \mathbf{b}_0}{|\Delta \mathbf{b}_0|^3}$$

where the cross product $\mathbf{a} \times \mathbf{b}$ is defined as

$$\mathbf{a} \times \mathbf{b} = a_x b_y - a_y b_x$$

we get the following:

$$\lambda = \sqrt{\frac{1}{2} \frac{\mathbf{t}_0 \times (\mathbf{b}_2 - \mathbf{p})}{\kappa_0}}$$

The above equation also introduces a geometric constraint on the control point $\mathbf{b}_2$: depending on the sign of the curvature, $\mathbf{b}_2$ has to lie above or under the tangent line. Within the given half-plane we are free to position $\mathbf{b}_2$ anywhere. Positioning the last control point on the tangent line will result in zero curvature.

If we'd like to set $\mathbf{b}_2$ we need a heuristic. There are several considerations, that one can take into account when designing such a heuristic. Using a construction for $\mathbf{b}_2$ that is *invariant under a specific operation* can exhibit advantages.

For example setting $\mathbf{b}_2$ as the closest point of the osculating circle $(\mathbf{p}_0, \kappa_0)$ to the osculating circle $(\mathbf{p}_1, \kappa_1)$, we get a construction that is *invariant under offsetting*, however, if the distance between $\mathbf{p}_0$ and $\mathbf{p}_1$ becomes significantly smaller than the radii of the osculating circles, this construction leads to unwanted loops.

To avoid these loops, let us use a different heuristic, which consists of setting $\mathbf{b}_2$ to be the closest point of the osculating circle $(\mathbf{p}_0, \kappa_0)$ to $\mathbf{p}_1$. This is not invariant under offsetting, but it allows the points to be interpolated to get arbitrarily close to each other, independent of the osculating circle radii.

Once we have the two local solution curves $\mathbf{c}_0(t)$ and $\mathbf{c}_1(t)$, we elevate both to degree 5 and use the appropriate 3-3 control points (the ones that have effect on the position, tangent and curvature computation at the required endpoint of the curve) from them to form a single quintic transition between $\mathbf{p}_0$ and $\mathbf{p}_1$ which satisfies (2.1).

2.2.2. Rational quadratic local solution

Using rational quadratics we can exactly describe the arc between the heuristically selected point of the osculating circle and the given endpoint [4].

Let us use the second heuristic scheme, and we will use the circular arcs as the local solutions for the interpolation problem.

If we'd like to maintain the positivity of the weights, higher degree rational Bézier curves might become necessary if the central angle of the arc becomes greater than or equal to π.

We can arrive at a global solution between $\mathbf{p}_0$ and $\mathbf{p}_1$ the same way as in the integral case: after degree elevating the curves to degree five, we only keep 3-3 control points and their weights from the local solutions and use them to form a rational quintic.

2.3. Direct solution to the interpolation problem

Let us now construct a direct, global solution to the interpolation problem (2.1).

2.3.1. Integral cubic solution

The planar integral cubic Bézier curve has 8 scalar degrees of freedom, which is exactly as many as we need to solve our point-tangent-curvature interpolation problem directly.

The interpolation condition of the positions $\mathbf{p}_0, \mathbf{p}_1$ and the tangents $\mathbf{t}_0, \mathbf{t}_1$ poses the following constraint on the control points $\mathbf{b}_i \in \mathbb{E}^2 (i = 0, ..., 3)$ of the integral cubic:

$$\begin{aligned}
\mathbf{b}_0 &= \mathbf{p}_0 \\
\mathbf{b}_1 &= \mathbf{p}_0 + \lambda \mathbf{t}_0 \\
\mathbf{b}_2 &= \mathbf{p}_1 - \gamma \mathbf{t}_1 \\
\mathbf{b}_3 &= \mathbf{p}_1
\end{aligned}$$

for some displacements $\lambda, \gamma > 0$ along the tangent lines at the endpoints. Solving for the unknown λ and γ we get the following system of nonlinear equations

$$\lambda^2 = \frac{2}{3\kappa_0}(\mathbf{t}_0 \times (\mathbf{p}_1 - \mathbf{p}_0 - \gamma \mathbf{t}_1)) \tag{2.2}$$

$$\gamma^2 = \frac{2}{3\kappa_1}(\mathbf{t}_1 \times (\mathbf{p}_0 - \mathbf{p}_1 + \lambda \mathbf{t}_0)) \tag{2.3}$$

In general, the above system does not have real positive solutions. In [1] de Boor et. al. proved, that if the data correspond to a smooth curve with non-vanishing curvature and if the distance between the interpolation points is sufficiently small, then the interpolation problem has positive real roots and it provides a high-accuracy approximation:

Theorem 2.1. *If* $\mathbf{f}$ *is a smooth curve with non-vanishing curvature and* $(\mathbf{b}_i, \mathbf{t}_i, \kappa_i)$, $(i = 0, 1, ...)$ *are position, tangent and curvature values of* $\mathbf{f}$ *at parameters* t_i, *and*

$$h := \sup_i |\mathbf{b}_{i+1} - \mathbf{b}_i|$$

and positive real solutions exist to (2.2, 2.3) on each segment between $\mathbf{b}_i$ *and* $\mathbf{b}_{i+1}$, *then the corresponding interpolant(s) satisfy*

$$dist(\mathbf{f}, \mathbf{b}_f) = O(h^6)$$

For proof and more details on the geometric constraints, please refer to [1]. This interesting result on the approximation order shows an advantage of interpolating geometrical quantities instead of parametrization dependent Hermite-type input data $\mathbf{r}, \mathbf{r}', \mathbf{r}''$.

2.3.2. Integral quartic solution

An integral quartic has 10 degrees of freedom. In order to satisfy position and tangent direction constraints, the control points $\mathbf{b}_i \in \mathbb{E}^2 (i = 0, ..., 4)$ must be set to

$$\begin{aligned} \mathbf{b}_0 &= \mathbf{p}_0 \\ \mathbf{b}_1 &= \mathbf{p}_0 + \lambda \mathbf{t}_0 \\ \mathbf{b}_2 & \\ \mathbf{b}_3 &= \mathbf{p}_1 - \gamma \mathbf{t}_1 \\ \mathbf{b}_4 &= \mathbf{p}_1 \end{aligned}$$

Let us now assume that $\mathbf{b}_2$ is given. We get the following for the $\lambda > 0, \gamma > 0$ displacements:

$$\begin{aligned} \lambda &= \sqrt{\frac{3}{4\kappa_0}(\mathbf{t}_0 \times (\mathbf{b}_2 - \mathbf{p}_0))} \\ \gamma &= \sqrt{\frac{3}{4\kappa_1}(\mathbf{t}_1 \times (\mathbf{b}_2 - \mathbf{p}_1))} \end{aligned}$$

This also restricts the position of the middle control point to be within one of the quarter planes determined by the two tangent lines. The actual quarter in which the control point has to be put depends on the sign of the curvatures at the endpoints.

3. Approximating offset curves

3.1. Approximation algorithm with PNC interpolation

We used the following algorithmic framework for offset curve approximation:

- Let us transform the endpoint data of the curve according to the amount of offset. This gives us our first approximation $\mathbf{r}_d^{(0)}(t)$ of the offset
- At each iteration, until a prescribed error threshold is reached, insert a new point, tangent and curvature triplet at the place of greatest error between the actual offset and the offset approximation $\mathbf{r}_d^{(i)}(t)$

The first approximation can be obtained by transforming the original curve's endpoint position, tangent and curvature data. Using the Frenet-Serret formulas for arbitrary parametric speed we can find

$$\mathbf{t}_d(t) = \operatorname{sgn}(1 - d\kappa(t))\mathbf{t}(t)$$
$$\kappa_d(t) = \frac{\kappa(t)}{1 - d\kappa(t)}$$

where $\operatorname{sgn}(x)$ is the signum function and d is the amount of offset. According to these, the original curve's endpoint data $(\mathbf{p}_i, \mathbf{t}_i, \kappa_i)$, $(i = 0, 1)$ has to be transformed as follows

$$\tilde{\mathbf{p}}_i^d = \mathbf{p}_i + d\mathbf{n}_i$$
$$\tilde{\mathbf{t}}_i^d = \operatorname{sgn}(1 - d\kappa_i)\mathbf{t}_i$$
$$\tilde{\kappa}_i^d = \frac{\kappa_i}{1 - d\kappa_i}$$

Using any of the previously discussed direct or indirect PNC interpolation methods, we get an approximation curve segment.

The recursive insertion results in a sequence of PNC triplets $d_i = (\mathbf{p}_i, \mathbf{t}_i, \kappa_i)$ $(i = 0, \ldots)$. The sequence of curve segments defined between triplets d_i and d_{i+1} form a G^2 spline.

3.2. Test results

The following table summarizes some test results. We measured the number of required PNC segments to reach two relative error limits, 10^{-2} and 10^{-5} for a cubic, a quartic and a rational quadratic curve.

	Integral quintic	Rational quintic	Integral quartic
Cubic 10^{-2}	4	3	5
Cubic 10^{-5}	18	10	77
Quartic 10^{-2}	4	4	9
Quartic 10^{-5}	20	12	65
Circular arc 10^{-2}	4	1	5
Circular arc 10^{-5}	17	4	79

Table 1: Test results.

As expected, the rational quintic indirect solution performed the best, followed by the integral quintic indirect solution. Due to the heuristic, the quartic scheme performed poorly. The rational quintic's need for 4 segments to get below the relative error 10^{-5} when approximating a circular arc, suggests that the heuristic used for the construction can be improved.

4. Summary

The theoretical results on exact representation of offsets of Farouki et. al. and Pottmann et. al. has been briefly reviewed. Easy to compute solutions for a point-tangent-curvature interpolation problem have been given and they have been used to approximate offset curves. Future work includes the more careful investigation of heuristics and their effect on the approximation order, as well as generalization to surfaces using similar geometric quantities as input.

References

[1] C. de Boor, K. Höllig, M. Sabin High accuracy geometric Hermite interpolation, 1987, Computer Aided Geometric Design, Volume 4, Issue 4, Pages 269–278.

[2] R. T. Farouki Pythagorean-hodograph curves in practical use, 1992, Geometry Processing for Design and Manufacturing (R. E. Barnhill, ed.), SIAM, 3–33.

[3] R. T. Farouki, J. Manjunathaiah, D. Nicholas, G.-F. Yuan, S. Jee Variable feedrate CNC interpolators for constant material removal rates along Pythagorean-hodograph curves, 1998, Computer Aided Design, ISSN 0010-4485, Vol. 30, Number 8, Pages 631–640.

[4] L. Piegl, W. Tiller The NURBS book (2nd ed.), 1997, ISBN 3-540-61545-8, Springer-Verlag New York, Inc.

[5] H. Pottmann Rational curves and surfaces with rational offsets, 1995, ISSN 0167-8396, Computer Aided Geometric Design, Vol. 12, Number 2, Pages 175–192.

Gábor Valasek, **János Vida**
Eötvös Loránd University, Faculty of Informatics
1117 Budapest, Pázmány Péter sétány 1/c.