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Interesting Surfaces in Nil Space*

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Abstract

W. Heisenberg's real matrix group provides a noncommutative translation group of an affine 3-space. The **Nil**-geometry, which is one of the eight Thurston 3-geometries, can be derived from this group. It was proved by E. Molnár in [M97] that the maximal simply connected homogeneous Riemannian 3-geometries have a unified interpretation in the 3-dimensional projective spherical space that can be embedded into the Euclidean 4-space.

Analogous to the Euclidean geometry we introduce the notion of the geodesic cone and torus in **Nil** geometry. We also show a visualization of the lattice-like optimal packing of the geodesic balls determined by the second author in [Sz06]. The pictures and animations were made by using the Wolfram Mathematica software.

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MSC: 53A35, 52C17, 52C22

1. On Nil geometry

The **Nil** geometry can be derived from the famous real matrix group $\mathbf{L}(\mathbb{R})$, discovered by Werner Heisenberg. The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+x & c+xb+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{pmatrix} \tag{1.1}$$

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defines "translations" $\mathbf{L}(\mathbb{R}) = \{(x,y,z) : x,y,z \in \mathbb{R}\}$ on the points of the space $\mathbf{Nil} = \{(a,b,c) : a,b,c \in \mathbb{R}\}$. These translations are not commutative in general.

The matrices $\mathbf{K}(z) \triangleleft \mathbf{L}(\mathbb{R})$ of the form

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to (0,0,z) \tag{1.2}$$

constitute the one parametric centre, i.e. each of its elemenst commutes with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called *fibre translations*.

The $\mathbf{Nil}$ geometry can be projectively interpreted by the "right translations", as the following matrix formula shows, according to (1.1).

$$(1;a,b,c) \longrightarrow (1;a,b,c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \tag{1.3}$$

The detailed description can be found in article [2].

In [4] Emil Molnár has shown, that a rotation through angle β about the z-axis at the origin, will be a quadratic mapping as follows:

$$\begin{aligned} &\overline{x} = x\cos(\beta) - y\sin(\beta),\ \overline{y} = x\sin(\beta) + y\cos(\beta), \\ &\overline{z} = z - \frac{1}{2}xy + \frac{1}{4}(x^2 - y^2)\sin(2\beta) + \frac{1}{2}xy\ \cos(2\beta). \end{aligned} \tag{1.4}$$

This is an isometry of $\mathbf{Nil}$, kééping invariant the infinitezimal Riemann metric (arc-length-square) $(ds)^2 = (dx)^2 + (dy)^2 + (-x\,dy + dz)^2$.

Figure 1 shows the path of the point $(1;2,1,1)$ when it is rotated about the z-axis.

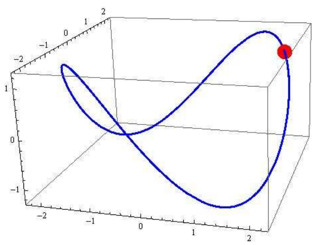

Figure 1: The point $(1;2,1,1)$ rotated about the z-axis

The geodesic curves of $\mathbf{Nil}$ geometry are generally defined as having locally minimal arc length between their any two (near enough) points. Because of the

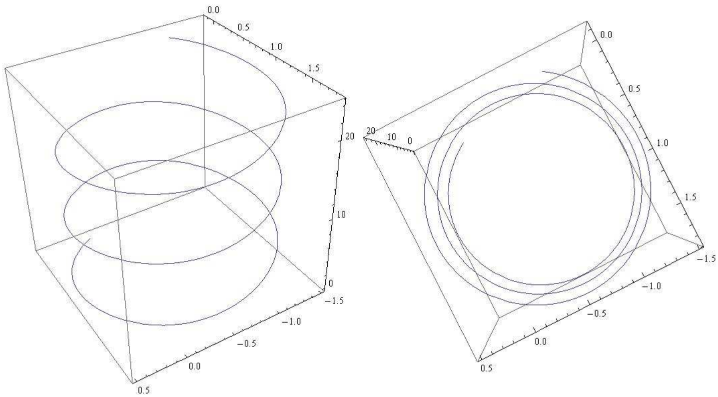

Figure 2: A geodesic curve, with $\alpha = \frac{\pi}{6}, \theta = \frac{\pi}{4}$

homogeneity of **Nil** geometry we can assume, that the starting point of an arbitrary geodesic curve is the origin, with initial values α and θ.

$$\begin{aligned} x(0) = y(0) = z(0) = 0;\ \dot{x}(0) = c\cos\alpha,\ \dot{y}(0) = c\sin\alpha \\ \dot{z}(0) = w;\ -\pi \leq \alpha \leq \pi. \end{aligned} \tag{1.5}$$

Tha arc length parameter is

$$s = \sqrt{c^2 + w^2}\, t,\ \text{where } w = \sin\theta,\ c = \cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \tag{1.6}$$

i.e. unit velocity can be assumed. The equation system of a helix like geodesic curve is:

$$\begin{aligned} x(t) = \frac{2c}{\omega}\sin(\frac{\omega t}{2})\cos(\frac{\omega t}{2} + \alpha),\ y(t) = \frac{2c}{\omega}\sin(\frac{\omega t}{2})\sin(\frac{\omega t}{2} + \alpha) \\ z(t) = \omega t \left\{1 + \frac{c^2}{2\omega^2}\left[1 - \frac{\sin(\omega t)}{\omega t} + \frac{1 - \cos(2\omega t)}{\omega t}\sin(\omega t + 2\alpha)\right]\right\}. \end{aligned} \tag{1.7}$$

if $w \neq 0$. In the case, when $w = 0$, the curve is the following parabola:

$$x(t) = ct\cos(\alpha), y(t) = ct\sin(\alpha), z(t) = \frac{1}{2}c^2t^2\cos(\alpha)\sin(\alpha). \tag{1.8}$$

The trivial solution will be $x(t) = y(t) = 0,\ z(t) = t$ if $w = 1,\ c = 0$.

Definition 1.1. The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the geodesic curve from P_1 to P_2.

2. Geodesic Cone

Using the **Nil** rotation and the geodesic line defined above, we can construct, an object called geodesic "cone" similarly to Euclidean geometry.

Definition 2.1. Let $g(t)$ be a geodesic line with starting point at the origin, with parameter $\theta \in \left[0, \frac{\pi}{2}\right)$ and $\alpha \in [0, 2\pi)$ (see (1.7)(1.8)). The geodesic cone C_θ is a surface of revolution, generated by revolving the given geodesic curve about the z-axis, where the **Nil** rotation is given by (1.4), and $t \in [0, \frac{2\pi}{\sin\theta}]$.

The geodesic curve returns periodically to the z-axis, so if we rotate a "whole" curve, then we get an object with self intersections. Because of this, we only rotate the part of the geodesic curve between the origin and the first return.

On the first picture of Figure 3 we can the case, if we rotate the whole geodesic curve. The second and third pictures show two "good" geodesic cones.

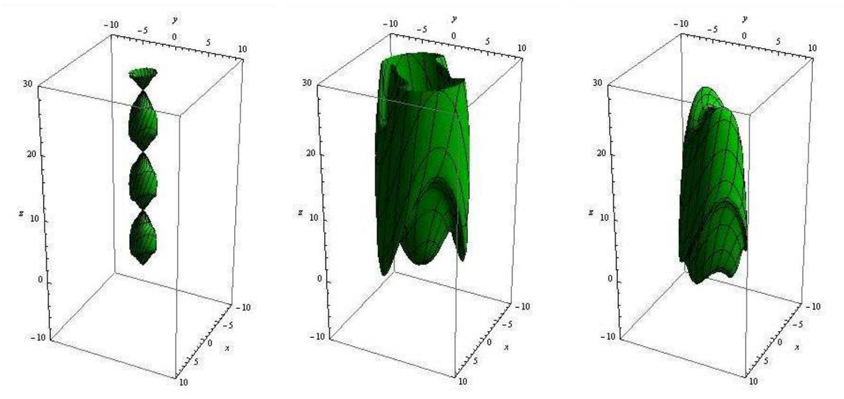

Figure 3: A "bad" cone, and two good ones

3. Geodesic Sphere

Definition 3.1. The geodesic sphere of radius R with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1, P_2) = R$. Moreover we require that the geodesic sphere is a simply connected surface without self-intersection in **Nil** space.

Tipically we chose the origin as the center of the sphere and ball, by the homogeneity of **Nil**. In [3] the following theorem was proven.

Theorem 3.2. *The geodesic sphere and ball of radius R exists in the **Nil** space if and only if $R \in [0, 2\pi]$.*

We could construct the sphere, by definition, using the geodesic line, but there is another method. The intersection of the sphere with the $[xz]$-plane is easily computable. By rotating this curve about the z-axis we can obtain the geodesic sphere in a more easily computable way. On the first two pictures of Figure 4 we can see this curve and the sphere obtained from the curve if the radius is 4.

The Theorem 3.2 states, that there are no geodesic spheres, with larger radius than 2π. On the two pictures on the right side of Figure 4 we can see the curve to be rotated and the so obtained sphere with "too large" radius ($R = 13$). We can see, that it has self intersections, which is the reason of the radius-condition in the 3.1 definition.

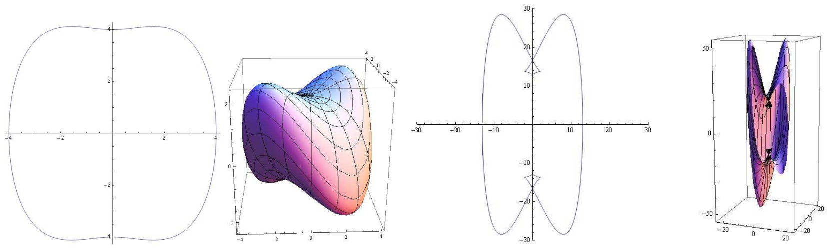

Figure 4: The curve to be rotated, and the so obtained geodesic sphere - with $R = 4$ on the left, and $R = 13$ (a wrong example) on the right

In [3] the second author examined the convexity of the geodesic ball in Eucledian sense in our affine model and obtained the following theorem (see picture 5):

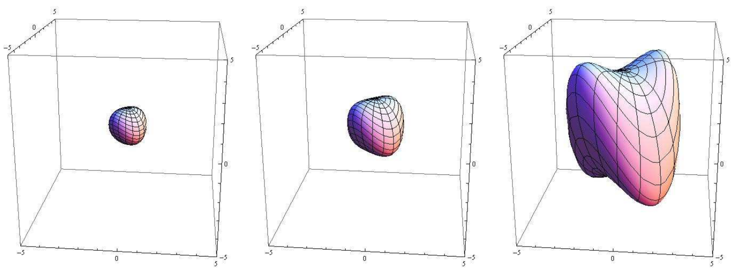

Figure 5: Various geodesic balls

Theorem 3.3. *The geodesic* ***Nil*** *ball is convex in affine-Eucledian sense in our model if and only if* $R \in [0, \frac{\pi}{2}]$.

On Figure 5 we illustrate the Theorem 3.3. On the first picture there is a ball with $R < \frac{\pi}{2}$, which is an example of an affine-Eucledian-convex ball. On the second picture is a ball with $R = \frac{\pi}{2}$, and on the third one a ball with $R > \frac{\pi}{2}$, which is non-convex in affine-Eucledian sense in our model.

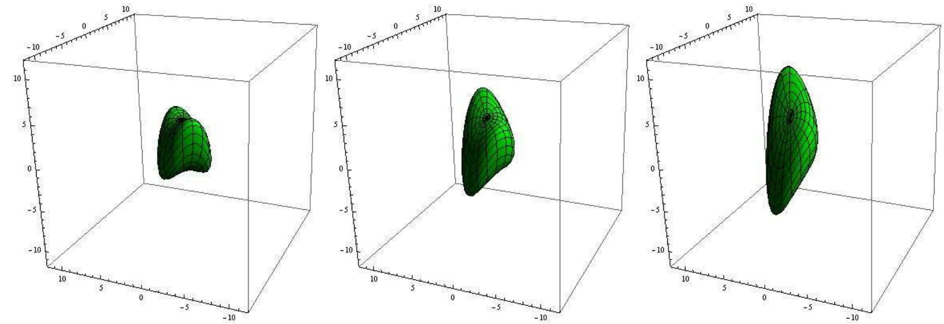

Figure 6: A geodesic ball at the origin, and its translated pictures

A geodesic sphere can be translated using the **Nil** translation defined in (1.1). It is easy to see, that translating into the x-direction is distorting the sphere. The pictures on Figure 6 are an example of this.

The geodesic balls will play a huge role in Section 5, in which we will visualize the optimal lattice like ball packing of the **Nil** geometry by a type of **Nil** lattices.

4. The Nil Torus

Definition 4.1. Let S be a geodesic sphere of radius $R \in [0, 2\pi]$,with a centre on the $[x, z]$ plane, and without intersection with the z-axis. The intersection of S with the $[x, z]$ plane is denoted by $S_{[x,z]}$. The *geodesic torus* T_S is a surface of revolution generated by revolving the given curve $S_{[x,z]}$ about the z-axis where the **Nil** rotation is given by formula (1.4).

Figure 7 shows some examples of the **Nil** tori. The third picture shows a torus translated by the **Nil** translation. We can see, that this translation destorts the torus, in a similar way to the sphere.

5. The densest lattice-like geodesic ball packing by a type of Nil lattices

In [3] the second author determined the densest lattice like geodesic ball packing by a type of **Nil** lattices. Using the tools above, we can visualize this construction. First we need to define the **Nil**-lattice.

Let τ_1, τ_2 be **Nil** translations, and $(\tau_3)^k = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1$ with this commutator $(k \in \mathbb{N}, k > 0)$. Generate the discrete group $(<\tau_1, \tau_2>, k)$ denoted by $L(\tau_1, \tau_2, k)$.

Definition 5.1. The **Nil** point lattice $\Gamma_P(\tau_1, \tau_2, k)$ is a discrete orbit of point P in the **Nil** space under the group $L(\tau_1, \tau_2, k)$ with an arbitrary starting point P for all $(k \in \mathbb{N}, k > 0)$.

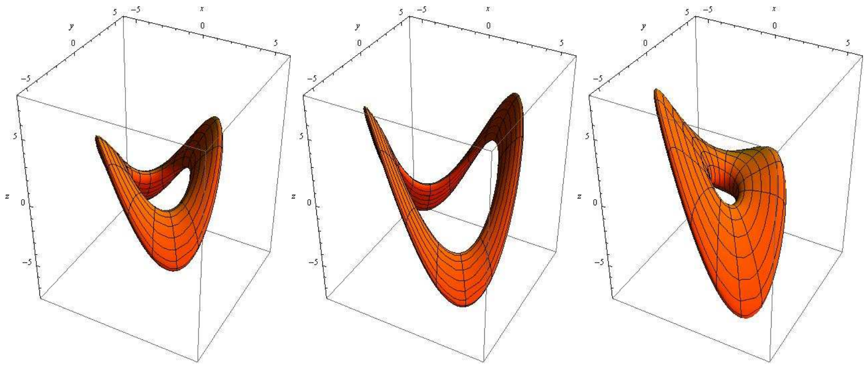

Figure 7: Geodesic tori

In case $k = 1$ according to [3] the optimal values of the T_1^{opt} and T_2^{opt}, which then generate the $\Gamma_P(T_1^{opt}, T_2^{opt}, k)$ lattice, for the optimal lattice-like ball packing:

$$\begin{gathered} t_1^{1,opt} \approx 1.3063382,\ t_1^{3,opt} = R_{opt} \approx 0.73894461; \\ t_2^{1,opt} \approx 0.6531691,\ t_2^{2,opt} \approx 1.13132206,\ t_2^{3,opt} \approx 1.10841692, \\ T_1^{opt} = (1, t_1^{1,opt}, 0, t_1^{3,opt}), T_2^{opt} = (1, t_2^{1,opt}, t_2^{2,opt}, t_2^{3,opt}) \end{gathered} \tag{5.1}$$

Using this values we can finally visualize the ballpacking. Figure 8 shows this construction. We note that the kissing number of the balls is 14, compared to the Euclidean case, which is 12.

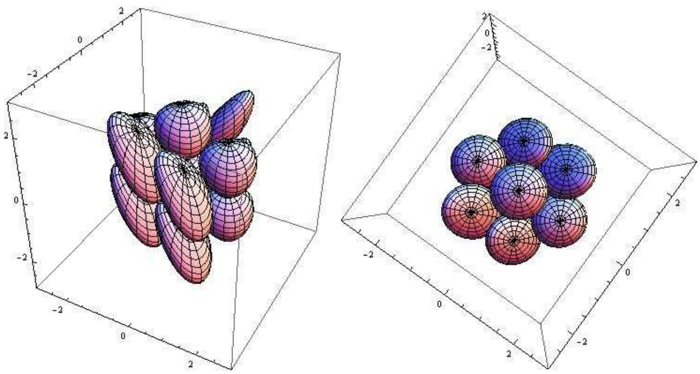

Figure 8: The optimal lattice-like geodesic ball packing in case $k = 1$

On Figure 9 we can see the same construction, but with "another layer" of balls around the ball with the origin as the center, as well as the centers of the balls on the picture.

Animations of the discussed surfaces are also available on the following site:
`http://demonstrations.wolfram.com/author.html?author=Benedek+Schultz`

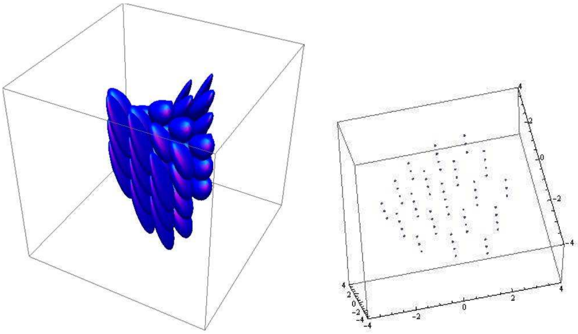

Figure 9: A picture with more balls, and the centers of the balls

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