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Sensitivity analysis of the energy functional defined on Sobolev space*

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Abstract

In the present work a numerical method is developed for the approximation of the $J \in (0, \infty)$ interval in Ricceri's theorem, to obtain the solution of elliptic type partial differential equations by using energy functionals. The critical surfaces are also approximated.

Keywords: energy functional, Sobolev space, Ricceri's theorem, critical points spline surface

MSC: 65D15, 35J20, 58E05.

1. Introduction

Let us consider the following partial differential equation with a boundary condition:

$$(P_\lambda) \begin{cases} -\triangle u = \lambda f(u), & x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a compact set.

The problem (P_λ) is a simplified form of certain stationary waves in the nonlinear Schrödinger equation, where the potential energy is zero, and the nonlinear term f is a perturbation, which satisfies the conditions in Ricceri theorem (Theorem 1.1). Under these conditions (P_λ) is a resonant problem.

We assign an energy functional $E_\lambda : H_0^1(\Omega) \to \mathbb{R}$ to this problem defined on Sobolev space $H_0^1(\Omega)$ given by:

$$E_\lambda(u) = \frac{1}{2}\|u\|^2_{H_0^1} - \lambda \int_\Omega F(u(x))\,dx$$

where

$$F(s) = \int_0^s f(x)\,dx.$$

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We know that E_λ is a continuous derivable and the critical points of E_λ are the weak solutions of the problem (P_λ).

The numerical calculation of the critical points is based on special case of the Ricceri's theorem [1]:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, with smooth boundary, and $f : \mathbb{R} \to \mathbb{R}$ a continuous function with $\sup_{x\in\mathbb{R}} \int_0^x f(t)\,dt > 0$. Assume that there a, q, s, γ, with $q < \frac{n+2}{n-2}$ (if $n > 2$), $s < 2$ and $\gamma > 2$, such that*

$$\begin{aligned} |f(x)| &\leqslant a\left(1+|x|^q\right) \ \forall x \in \mathbb{R}, \\ \int_0^x f(t)\,dt &\leqslant a\left(1+|x|^s\right) \ \forall x \in \mathbb{R} \end{aligned}$$

and

$$\limsup_{x\to 0} \frac{\int_0^x f(t)\,dt}{|x|^\gamma} < +\infty.$$

Then there exists an open interval $J \subseteq [0,+\infty[$ such that for each $\lambda \in J$ the problem (P_λ) has at least three distinct weak solutions in $H_0^1(\Omega)$.

In the present work a numerical method is developed for the determination of the $J \subset (0,\infty)$ interval for concrete problems.

2. The numerical method

The basic idea is: We approximate the directional derivate of E_λ for certain u elements (smooth surfaces) in du direction:

$$\frac{dE_\lambda}{du}(u) = \lim_{t\to 0} \frac{E_\lambda(u+tdu) - E_\lambda(u)}{t}.$$

The critical points are the solutions of the equation:

$$\frac{dE_\lambda}{du}(u) = 0.$$

The solutions of the equation can be approximated with the solutions of the following equation by taking small values for $t \neq 0$

$$E_\lambda(u+tdu) - E_\lambda(u) = 0, \ \forall du.$$

If we consider a u surface (smooth) in $H_0^1(\Omega)$, then by resolving the previous equation, we obtain

$$\lambda = \frac{1}{2} \frac{\|u+tdu\|^2 - \|u\|^2}{\int\limits_\Omega \left[F(u(x)+tdu(x)) - F(u(x))\right]dx}. \tag{2.1}$$

By using formula (2.1), we calculate the value of λ where u could be a critical surface for E_λ. We proceed from a little modified surface u_0.

We consider a compact set Ω in $\mathbb{R}^2$, and a sublinear function $f = f(u)$.

The algorithm of determination of λ:

Step 1. We take a grid with step h on Ω. We consider the values of the surfaces in the intersections: $u(x_i, y_j) = u_{i,j}$

Step 2. We approximate u with cubic spline surfaces where $(x_i, y_j, u_{i,j})$ are the control points.

Step 3. We approximate $(\nabla u)_{i,j}$ in the interior of Ω by using the following formulas of approximations at second degree

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \quad \left(\frac{\partial u}{\partial y}\right)_{i,j} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2h},$$

and at the margin by using

$$\left(\frac{\partial u}{\partial x}\right)_{i,j} \approx \frac{u_{i\pm1,j} \mp u_{i,j}}{\pm h}, \quad \left(\frac{\partial u}{\partial y}\right)_{i,j} \approx \frac{u_{i,j\pm1} \mp u_{i,j}}{\pm h}.$$

Step 4. We count out the integrates by using the trapezoid rule.

$$\|u\|^2_{H^1_0} = \int_\Omega |\nabla u|^2 \, dx, \text{ and } \int_\Omega F(u(x)) \, dx.$$

Step 5. Let $u := u + t \cdot du$.

Step 6. We apply steps 3 and 4 for a new value at u.

Step 7. We calculate the value of λ with formula (2.1).

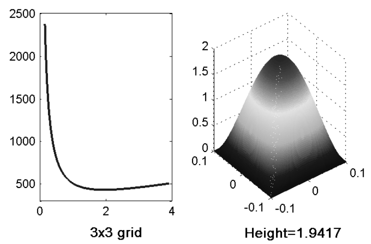

Figure 1: Graph of λ for 3×3 grid

3. Results

By using the Matlab program, we studied the variations of λ for the $f(u) = \arctan^2(u)$ sublinear function, where Ω is a square.

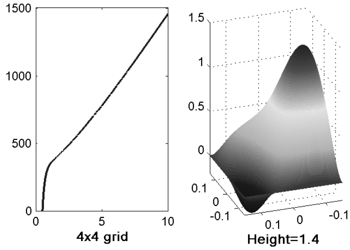

Figure 2: Graph of λ for 4×4 grid

We take a 3×3 grid in Ω. The nodes of the grid are the control points of the surface. If we change the value of $u_{2,2}$, the control point in the middle of the grid, between 0 and 4, and the values of the other control points remain the same, we obtain Figure 1.

The Ox axis indicates the height of the altering control point, and the Oy axis gives the λ values. The graph shows that on which λ value the framed surface is the critical surface of the energy functional, for a given height of the control point.

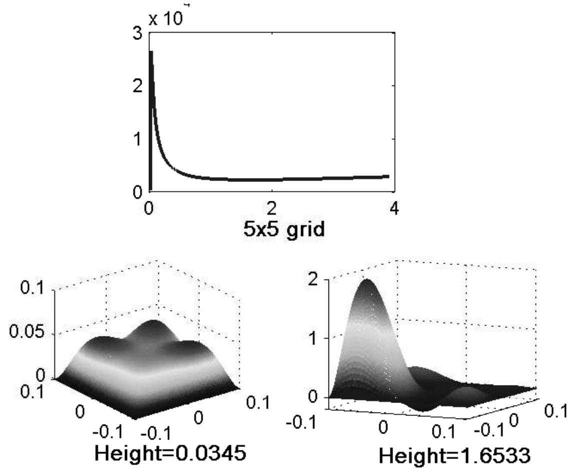

Figure 3: Graph of λ for 5×5 grid

The surface given by Figure 1 presents the surface when the graph has local minimum

If we consider four control points in the interior of Ω (4×4 grid), where three is fixed and the value of the forth point varies, then we obtain Figure 2. The surface beside the graph is the surface where the graph has inflection point.

If we consider nine control points in the interior of Ω (5×5 grid), where the value in one of the extreme control points varies, and the values of the remaining eight control points are fixed, then we obtain Figures 3. The surfaces under the graph represent the surfaces where the graph of Figure 3 has local maximum and local minimum, respectively.

If we consider again nine control points in the interior of Ω (5×5 grid), where we vary the height of the control point in the middle of the grid, we obtain Figures 4. The surfaces under the graphs represent the surfaces where the graphs of Figure 3 have vertical asymptote, local maximum and local minimum, respectively.

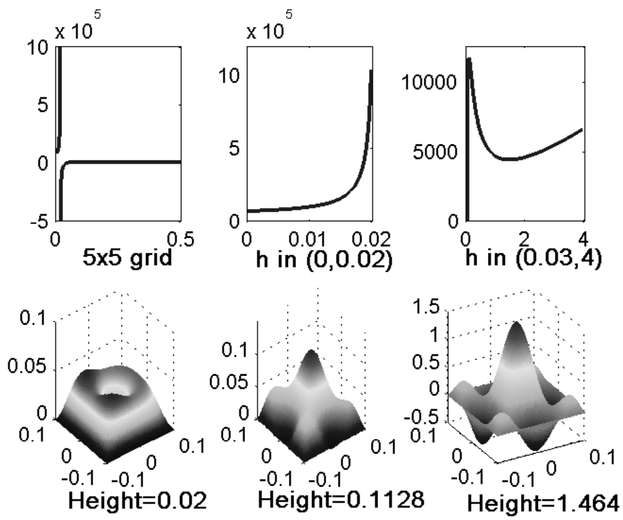

Figure 4: Graph of λ for 5×5 grid

We have to examine the correctness of our method, because approximation errors could intervene. That is by taking lower steps in the derivation and integration formulas we have to obtain better approximation results. Therefore we obtain a converging array of curves for different kinds of resolution, that is every time we divide the resolution in two equal parts (see Figure 5).

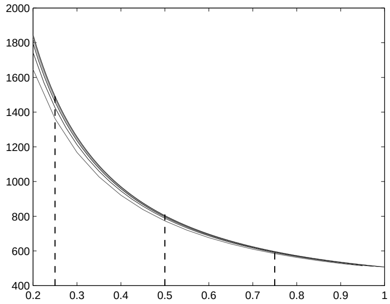

Figure 5: Convergence

Table 1 shows the distance between two consequent graphs for given resolutions and height of control points. Accordingly, it is suggested to take small resolution where the little change of the control point height induces big changes of the λ value.

	0.25	**0.5**	**0.75**
0.05-0.025	63.31	15.33	6.06
0.025-0.0125	33.51	7.92	3.12
0.0125-0.00625	17.23	4.03	1.58
0.00625-0.003125	8.73	2.03	0.8
0.003125-0.0015625	4.39	1.02	0.4
0.0015625-0.00078125	2.21	0.51	0.2

Table 1: Converging distances

4. Conclusions

We can approximate the $J \subset (0, \infty)$ interval in the Ricceri's theorem with the presented method. We can approximate the interval of λ with 0 and 27000 by examine the obtained figures. So the case is worth dealing with is when we take values between 0 and 27000 for λ.

Also we can determine the values in the control points for which we can find critical surfaces of E_λ for a given λ. If, for example, we choose the value of λ equal to 5000 we obtain seven surfaces which could be critical values of the energy functional E_λ (see Figure 6). The altering control points height are written under the figures.

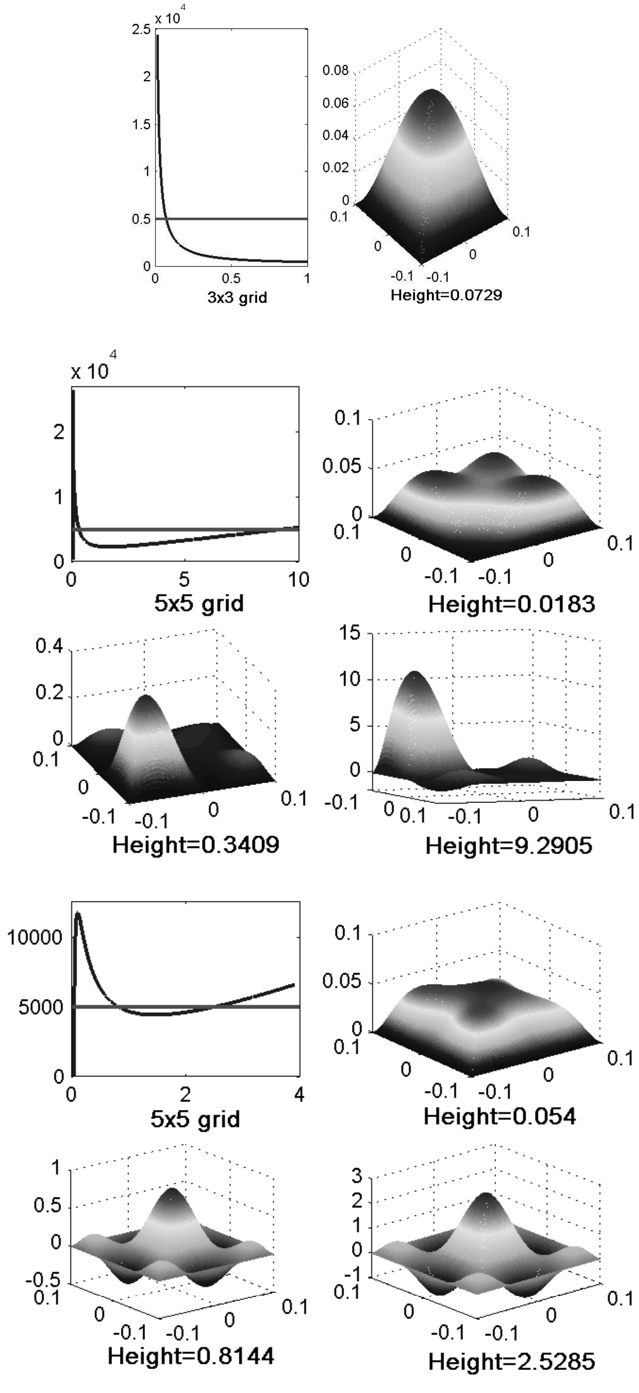

Figure 6: The possible critical surfaces for $\lambda = 5000$

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