Proceedings of the 7^{th} International Conference on Applied Informatics
Eger, Hungary, January 28–31, 2007. Vol. 1. pp. 45–53.

On fractions $\frac{y}{x}$ equal to their g-decimal representation $x + g^{-k}y$

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Abstract

Recently St. Deschauer noticed that both digits of the decimal fraction 2.5 occur in the equivalent reduced fraction 5/2, and he posed and solved the question to find all such numbers in [1]. In this paper we present another and perhaps shorter approach for his result using geometric interpretations of number theoretical facts. Thereby we do not focus on a cubic equation as in [1], but on hyperbolas of an osculating pencil of conics. This interpretation opens up for the "continuous case", which contains also the golden ratio as an example.

Keywords: Elementary Number Theory, Fractions, Elementary Geometry

MSC: 11Axx, 51Mxx

1. Introduction

For natural numbers x and y between 1 and 9 it is easy to test that, on the basis of the decimal positioning system the values $x = 2$, $y = 5$ form the only pair of numbers, which equals

$$\frac{y}{x} = x + \frac{y}{10}, \quad \left(\frac{5}{2} = 2.5\right).$$

Obviously x can neither be 0 nor 1 and to make sure that $\frac{y}{x} > 1$ the digits y and x have to fulfil the condition

$$y > x > 1. \tag{1.1}$$

A finite decimal fraction

$$z = x + \sum_{k=1}^{n} 10^{-k} \cdot y_k, \quad x \in \mathbb{N} \setminus \{1\}, \quad 0 \leqslant y_k \leqslant 9 \tag{1.2}$$

always can be seen as a number of type

$$z = x + g^{-1}y, \tag{1.3}$$

which is based on a generalized number system with $g = 10^n$ as the "g-decimal positioning unit".

In the following we think of such generalized g-decimal number systems, which are based on an arbitrarily given number $g \in \mathbb{N} \setminus \{1\}$ and its powers $q := g^k$, $k \in \mathbb{Z}$. We will call the condition

$$\frac{y}{x} = x + q^{-1}y, \quad x < y < q, \tag{1.4}$$

the DESCHAUER-condition of a q-decimal fraction.

2. Geometric interpretation of the Deschauer condition

The generalized q-decimal fraction

$$z = x + g^{-1}y, \quad x, y \in \mathbb{R}, \quad g, k \in \mathbb{N} \tag{2.1}$$

represents a plane φ in $\mathbb{R}^3$ while the quotient

$$z = \frac{y}{x} \tag{2.2}$$

represents a very special hyperbolic paraboloid $\Phi \in \mathbb{R}^3$. Therewith all solutions $\{(x, y)\}$ belong to the intersection $c := \varphi \cap \Phi$ and to the integer grid in $\mathbb{R}^2$. The orthogonal projection of c into the xy-plane gives the hyperbola h

$$x^2 + q^{-1}y - y = 0, \quad (x, y) \in \mathbb{R}^2, \quad q = g^k. \tag{2.3}$$

The discussion of h becomes somehow easier, when using the projective extension $P\mathbb{R}^2$ of $\mathbb{R}^2$ by the mapping

$$(x, y) \mapsto (1, x, y)\mathbb{R} = (x_0, x_1, x_2)\mathbb{R}, \quad x = \frac{x_1}{x_0}, \quad y = \frac{x_2}{x_0}.$$

Now the polarity π to h is represented by the symmetric matrix

$$A = \begin{pmatrix} 0 & 0 & -q \\ 0 & 2q & 1 \\ -q & 1 & 0 \end{pmatrix}, \quad \text{resp. } A^{-1} = \begin{pmatrix} -1 & -q & q^2 \\ -q & q^2 & 0 \\ 2q^2 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

Note that $\pi(A)$ maps points of $P\mathbb{R}^2$ to lines, while $\pi(A^{-1})$ maps lines of $P\mathbb{R}^2$ to points! The centre M of h is the $\pi(A^{-1})$-image of the line u at infinity, $(u \ldots x_0 = 0)$:

$$M = (1, q, -2q^2). \tag{2.5}$$

The points of h at infinity are

$$U = (0, 0, 1)\mathbb{R} \quad \text{and} \quad \bar{U} = (0, 1, -q)\mathbb{R}. \tag{2.6}$$

And the asymptotes a, $\bar{a}$ of h have equations

$$a \ldots x = q, \quad \bar{a} \ldots q^2 + qx + y = 0. \tag{2.7}$$

Figure 1 shows the characteristics of hyperbola h in a standard Cartesian frame.

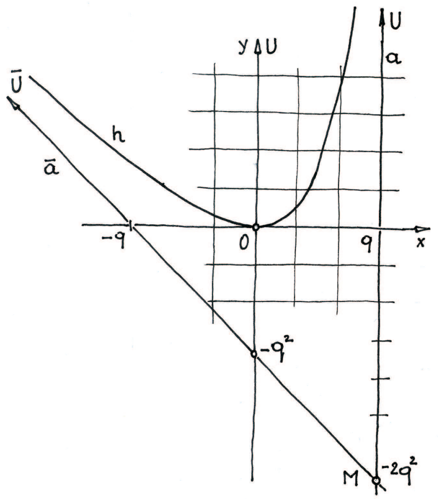

Figure 1: Hyperbola h to an arbitrarily given number q.

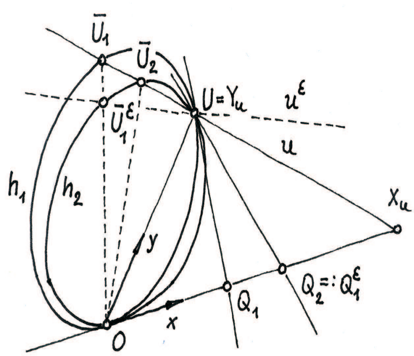

Figure 2: Projective geometric sketch of hyperbolas $h_1(q_1)$, $h_2(q_2)$.

Let $h_1(q_1)$, $h_2(q_2)$ be two hyperbolas belonging to the same g-decimal system, what means that $q_1 = g^{k_1}$; $q_2 = g^{k_2}$. Such hyperbolas h_1, h_2 have only $U = (0, 0, 1)\mathbb{R}$ and the origin $O = (1, 0, 0,)\mathbb{R}$ in common and they osculate in O, both touching the x-axis in this point. Therefore there must exist an elation $\varepsilon : h_1 \to h_2$ with centre O and the y-axis as axis of fixed points and the pair of related points $Q_1 = (1, g^{k_1}, 0)\mathbb{R}$ and $Q_2 = (1, g^{k_2}, 0)\mathbb{R}$.

Thus ε is described by the matrix

$$T = \begin{pmatrix} 1 & g^{-k_1}(g^{k_2-k_1} - 1) & 0 \\ 0 & g^{2(k_2-k_1)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.8}$$

Figure 2 shows a projective geometric sketch of a pair of osculating hyperbolas $h_1(q_1)$, $h_2(q_2)$. Obviously ε is not compatible with the integer grid!

3. Some consequences of the geometric interpretation

a) As the pairs (x, y) and (px, py), $p \in \mathbb{R} \setminus \{0\}$, define the same quotient z, the hyperbolic paraboloid Φ has "horizontal" (i.e. parallel to the xy-plane) generators, which all meet the z-axis of the (x, y, z)-frame. Thus the intersection of level lines $l(z)$ of Φ (2.2) and of a plane $\varphi(g^k)$ (2.1) is just a linear operation! From (2.1) follows that the level lines $l(z)$ of $\varphi(g^k)$ are parallel to the asymptote $\bar{a}$ of h. This motivates to construct hyperbola h according to J. Steiner by two projective pencils of lines: The first pencil has support O, the second one has support $\bar{U}$ at the line u at infinity. Figure 3 shows a projective sketch of this Steiner-generation of h as well as the Cartesian image of it. The projectivity $\rho : \{l \,|O \in l\} \to \{l' |\, \bar{U} \in l'\}$ is well-defined by three pairs (l, l'), namely

$$l_1 = OU \mapsto l_1' = \bar{U}U = u, \quad l_2 = OX_u \mapsto l_2' = \bar{U}O, \quad l_3 = O\bar{U} \mapsto l_3' = \bar{a}.$$

Furthermore we read of Figure 3 that the trilateral to any point is translatoric congruent to trilateral $\Delta(l_P, l_P', a)$. Denote the vertices of the first trilateral by P, Q, R and those of the other one by $\bar{Q}, O, \bar{R}$, we find that $\bar{R}$ is fixed and therefore $\mathrm{dist}(Q, O) = q^2 = \text{const.}$ for all $P \in h$.

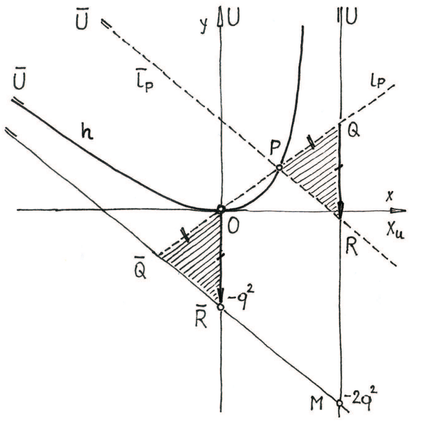

Figure 3: Projective sketch and Cartesian image of the Steiner-generation of h by projective pencils of lines.

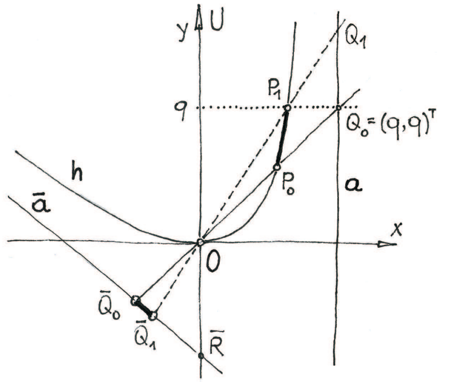

Figure 4: Segment of $\bar{a}$ which may contain proper solutions of the Deschauer-condition.

We also notice that the projectivity ρ induces a translation of the point-set of the asymptote a by the vector $\overrightarrow{QR}$ with $\|\overrightarrow{QR}\| = q^2 = g^{2k}$; and that this translation is compatible with the integer grid in $\mathbb{R}^2$. Now we focus on the integrity of point Q in Figure 3.

As it is must be possible to reduce the g-decimal fraction

$$z = \frac{g^k x + y}{g^k} \to z = \frac{y}{x}, \tag{3.1}$$

the number $q = g^k$ must be divisible by x. So we can conclude that, if point $P \in h$ has integer coordinates (x, y), so must Q also have integer coordinates. Is this the case, so it follows by the translation vector $\overrightarrow{QR}$, that $\bar{Q} \in \bar{a}$ has integer coordinates, too. Therefore we can state

Corollary 3.1. *Each proper solution (x, y) of the* DESCHAUER-*condition* (1.4) *necessarily stems from a point $\bar{Q}$ on the asymptote $\bar{a}$ of hyperbola h, which has integer coordinates. A proper solution (x, y) additionally has to fulfil the condition $1 < x < y < q$.*

Figure 4 shows the segment $[\bar{Q}_0, \bar{Q}_1] \subset \bar{a}$ containing the only points $\bar{Q}$, which correspond to eventually existing proper solutions of the DESCHAUER-condition.

b) Corollary 3.1 gives only necessary conditions for proper solutions. Sufficient would be, if we could answer the question: When does the line $\bar{Q}O$, $\bar{Q}$ with integer coordinates, hit the "vertical" asymptote a of h in an integer grid point? (See Figure 3 and Figure 4)

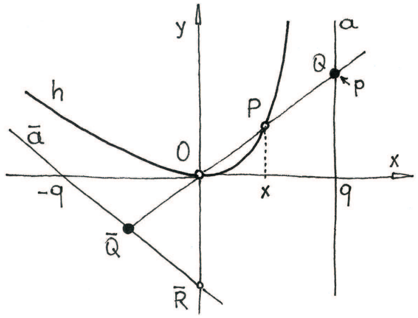

Figure 5: Relationship between the coordinates of $P, \bar{Q}, Q$.

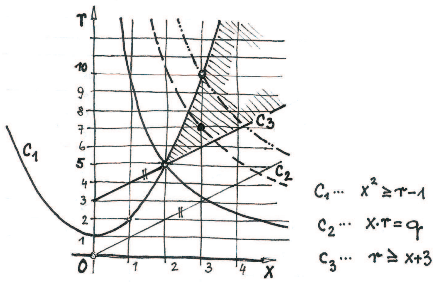

Figure 6: Arc of hyperbola c_2 to possible factor representations $xr = q$.

Let $P = (x, y)$ be an arbitrary point of hyperbola h, $P \neq O$. Then the corresponding point $\bar{Q} \in \bar{a}$ has the coordinates $(-q + x, -qx)$ and for point $Q \in a$ we therefore receive

$$Q = (q, s), \quad s = \frac{xq^2}{q - x}. \tag{3.2}$$

Because of assumption (3.1) we know that such that from (3.2) follows

$$s = \frac{x^2 r^2}{r - 1}. \tag{3.3}$$

As r^2 is not divisible by $r-1$, we find that $s \in \mathbb{N}$, iff $x^2|(r-1)$. So, in case x is prime, $r-1$ must be a square number. We collect these results in

Corollary 3.2. *A point $\bar{Q} \in [\bar{Q}_0, \bar{Q}_1] \subset \bar{a}$ with integer coordinates $(-q+x, -qx)$ and fulfilling the conditions of corollary 3.1 leads to a proper solution of the* DESCHAUER-*condition, if and only if*

$$q|x \wedge x^2| \left(\frac{q}{x} - 1\right).$$

c) In the following we will give better estimates fort the endpoints of the segment $[\bar{Q}_0, \bar{Q}_1] \subset \bar{a}$, see Figure 3. From condition $x < y$ we receive a limit point $\bar{Q}_0 = \left(-\frac{q^2}{1+q}, -\frac{q^2}{1+q}\right)$, which never can lead to a proper solution. The conditions

$$x^2 \geqslant r-1, \quad xr = q, \quad (x, r \in \mathbb{N} \setminus \{1\}) \tag{3.4}$$

restrict the area of possible solutions x according to Figure 6 and (3.4) to an arc of an equilateral hyperbola c_2 between the intersection with a parabola c_1 and a line of slope 1.

x	r_i	q_i	y_i
2	5	10	5
3	10	30	10
4	17; 9	68; 36	17; 18
5	26	130	26
6	37; 19; 13; 10	222; 114; 78; 60	37; 38; 39; 40
7	50	350	50
8	65; 33; 17	520; 264; 136	65; 66; 68
9	82; 28	738; 252	82; 84
10	101; 51; 26; 21	1010; 510; 260; 210	101; 102; 104; 105

Table 1: Possible basis $q(x)$ and corresponding values $y(q)$ to given abscissa x

Furthermore we conclude from (3.4).

Corollary 3.3. *The basis number q cannot be a prime number or a power of a prime number, but must be the product of at least two primes.*

Because of $x \geqslant 2$ we find $r \leqslant \frac{q}{2}$ and by (3.4) the smallest possible value r must be 5. Therefore we can state

Corollary 3.4. *The minimal basis number q, such that there exists a pair (x, y) fulfilling the* DESCHAUER-*condition in a g-decimal system is $q = g = 10$. In this 10-decimal system* (2.5) *is the only proper solution.*

By Corollary 3.2 and (3.4) we can construct the possible values q and y to a given value of x. Because of

$$x < y < q = rx \quad \wedge \quad y = \frac{rx^2}{r-1} \quad \Rightarrow \quad x < r - 1 \tag{3.5}$$

we have to consider all factorisations of x^2 with one factor $r_i > x + 1$. The table 1 shows the first cases:

Furthermore we deduce from (3.4) and Corollary 3.2 that the number q to a given value x can neither be too big or too small. The exact boundaries are

$$x(x+2) \leqslant q(x) \leqslant x^2(x+1). \tag{3.6}$$

Reversely, also x to a given q is bound by two extreme values

$$\mathrm{int}(\sqrt[3]{q}) \leqslant x(q) \leqslant \mathrm{int}(\sqrt{q}) - 1. \tag{3.7}$$

Obviously there exist infinitely many base-values q, as we can construct at least one such number to any given x.[1] We collect these and some more obvious results in

Corollary 3.5. *To a given value $x \in \mathbb{N}$, $(x \geqslant 2)$, there exists at least one basis-values $q(x) \in \mathbb{N}$ and $q \geqslant 10$. Therefore there are infinitely many such q, and because of $q = xr$, there never occur primes or powers of primes as numbers q.*
If x is prime, then $q = x(x^2+1)$ and $q = x(x^2+1)$ are unique.
If $x = x_1x_2$, x_j prime, then there exactly k cases $q_i(x)$ with $q_i := xr_i$, $r_i = x_1^{k+i} + 1$, $i = 1, \ldots, k$.
*If $x = x_1x_2$, x_j prime, $(x_1 < x_2)$, then there are exactly **four** cases $q_i(x)$.*
If $x = x_1x_2x_3$, x_j prime, $(x_1 < x_2 < x_3)$, then $\#\{q_i(x)\} \leqslant 24$.

4. Extensions of the Deschauer problem to reals

Until now we discussed problem to find triplets (x, y, q) of natural numbers fulfilling the Deschauer condition, which might be understood as the "Deschauer problem". In the following we extend this problem to real numbers.

Equation (2.3) describes hyperbola h as a conic in $\mathbb{R}^2$. So to an arbitrarily given basis q there is a continuous set of pairs (x, y) solving (2.3). For example, to $q = 10$, the periodic number $1,\bar{1}$ fulfils (2.3), when putting $x = 1$, $y = 1,\bar{1}$. For the triplet $(x, y, q) := (1,618\ldots, 1, -1)$, x the "Golden Number" τ, we receive the well-known fact that

$$1,618\ldots - 1 = \frac{1}{1,618\ldots} = 0,618\ldots$$

[1] In [1] this result is found by using another set of arguments. Also the result that q never can be a prime or a power of a prime can already be found in [1].

So, in an extended sense, also the Golden Number τ fulfils the Deschauer condition. Hyperbola h to $q = -1$ is represented in Figure 7.

Therewith V. de Spinadel's "Metallic Numbers" [2], [3], solving equations similar to (2.3) get an additional meaning: They are solutions of an extended Deschauer problem.

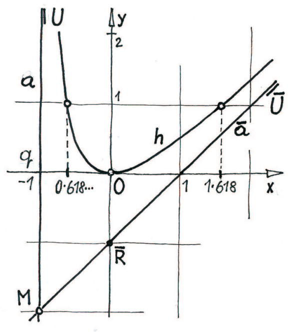

Figure 7: The Golden Number $x = \tau$ fulfils the Deschauer condition for $q = -1$, $y = 1$

5. Conclusion

The main goal of revisiting the topic introduced in [1] is to show that geometric interpretations of algebraic equations effectively support a number theoretical treatment. It remains open to discuss values $q(x)$ to which one can find more than one pair (x, y). Are there "many" such $q(x)$?

We did not discuss $q(x) = g^k$, $k > 1$, even we introduced such a treatment at the very beginning of that paper. An informatics approach would be needed, when big numbers occur in $(x, y, q(x))$. How to visualize the set $(x, y, q(x))$? To our opinion the presented topic is well suited as exercise material for different courses of mathematics or informatics. And there remain enough open problems to stimulate students to do research on there own.

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