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Approximation of solutions of SDE driven by multifractional Brownian motion

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Abstract

The aim of this paper is to approximate the solution of a stochastic differential equation driven by multifractional Brownian motion using a series expansion for the noise. We prove that the solution of the approximating equations converge in probability to the solution of the given equation.

Keywords: :Stochastic differential equations, approximation, multifractional Brownian motion.

1. Introduction

Let $t_0 \in (0, T]$ fixed. We consider the stochastic differential equation of the form driven by multifractional Brownian motion

$$\begin{aligned} dX(t) &= F(X(t), t)dt + G(X(t), t)dB(t), \\ X(t_0) &= X_0, \end{aligned} \tag{1.1}$$

where the random functions F and G satisfy with probability 1 the following conditions: $F \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n), G \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$ and for each $t \in [0, T]$ the functions $F(\cdot, t), \dfrac{\partial G(\cdot, t)}{\partial x}, \dfrac{\partial G(\cdot, t)}{\partial t}$ are locally Lipschitz. In [4] we gave an approximation for the fractional Brownian motion case. In real datasets the roughness of the sample path varies with location. Levy Vehel (1995) considered the case where the Hurst index varies with the time and named multifractional Brownian motion. In this paper we will approximate the solution in multifractional case using as Hurst index the linear and logistic functions:

$$H(t) = t \ \ and \ H(t) = 0.3 + \frac{0.3}{1 + exp(-100(t - 0.7))}.$$

The multifractional Brownian motion $B = \Big(B(t)\Big)_{t \in [0,1]}$ with Hurst index $H(t) \in (0, 1)$ we approximate using a trigonometrical series expansion.

Let J_ν be the Bessel function of first type of order ν and let $x_1 < x_2 < \cdots$ be the positive, real zeros of J_{-H}, while $y_1 < y_2 < \cdots$ are the positive, real zeros of J_{1-H}. We consider $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ to be two independent sequences of centered Gaussian random variables such that for each $n \in \mathbb{N}$ we have

$$\mathrm{Var}X_n = \frac{2c_H^2(t)}{x_n^{2H(t)} J_{1-H(t)}^2(x_n)}, \quad \mathrm{Var}Y_n = \frac{2c_H^2(t)}{y_n^{2H(t)} J_{-H(t)}^2(y_n)},$$

where

$$c_H^2(t) = \frac{\sin(\pi H(t))}{\pi}\Gamma(1+2H(t)).$$

In [2] it is proved that a fractional Brownian motion $B = \Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index $H \in (0,1)$ can be written as

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1].$$

Similarly on can shown that the multifractional Brownian motion $B = \Big(B(t)\Big)_{t\in[0,1]}$ with Hurst index $H(t) \in (0,1)$ can be written as

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1].$$

The equation (1.1) we approximate for each $N \in \mathbb{N}$ through

$$\begin{aligned} dX_N(t) &= \alpha(X_N(t),t)dt + \beta(X_N(t),t)dB_N(t), \\ X_N(t_0) &= X_0, \end{aligned} \tag{1.2}$$

where

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1], N \in \mathbb{N}.$$

We will show that the equation (1.2) has a local solution which converges in probability to the solution of (1.1) in the interval, where the solutions exist.

2. Series expansion for multifractional Brownian motion B

A Gaussian random process $B = \Big(B(t)\Big)_{t\geqslant 0}$ is called *multifractional Brownian motion with Hurst index* $H(t) \in (0,1)$, if it has zero mean, continuous sample paths and covariance function

$$E\Big(B(s_1)B(s_2)\Big) = \frac{1}{2}\Big({s_1}^{2H(t)} + {s_2}^{2H(t)} - |s_1 - s_2|^{2H(t)}\Big).$$

The multifractional Brownian motion B has on any finite interval $[0, T]$ Hölder continuous paths with exponent $\gamma \in (0, H(t))$ (see [1]), i.e.

$$\mathbb{P}\left(\omega \in \Omega : \sup_{\substack{0<s_2-s_1<h(\omega)\\ s_1,s_2\in[0,T]}} \frac{|B(s_2)-B(s_1)|}{|s_2-s_1|^\gamma} \leqslant \delta\right) = 1,$$

where h is an a.s. positive random variable and $\delta > 0$ is an appropriate constant. Moreover, the quadratic variation on $[a, b] \subseteq [0, T]$ is

$$\lim_{|\Delta_n|\to 0} \sum_{i=1}^{n} \left(B(t_i^n) - B(t_{i-1}^n)\right)^2 \quad = \quad \begin{cases} \infty & \text{if } H(t) < \frac{1}{2}, \\ b-a & \text{if } H(t) = \frac{1}{2}, \\ 0 & \text{if } H(t) > \frac{1}{2}, \end{cases} \tag{2.1}$$

where $\Delta_n = (a = t_0^n < \cdots < t_n^n = b)$ is a partition of $[a, b]$ with $|\Delta_n| = \max\limits_{1\leqslant i\leqslant n} (t_i^n - t_{i-1}^n)$.

If $H(t) \neq \frac{1}{2}$, then the convergence in (2.1) holds with probability 1 uniformly in the set of all partitions of $[a, b]$, while for $H(t) = \frac{1}{2}$ the convergence in (2.1) holds in mean square uniformly in the set of all partitions of $[a, b]$. Note that, if $H(t) \neq \frac{1}{2}$, then B is not a semimartingale, so the classical stochastic integration does not work. But the Hölder continuity of B will ensure the existence of integrals

$$\int_0^T G(u)dB(u),$$

defined in terms of fractional integration (see Section 4) as investigated in [9] and [10].

For $\nu \neq -1, -2, \ldots$ *the Bessel function* J_ν *of the first type of order* ν is defined on the region $\{z \in \mathbb{C} : |\arg z| < \pi\}$ as the absolutely convergent sum

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{\nu+2k}.$$

It is known that for $\nu > -1$ the function J_ν has a countable number of real, positive simple zeros (see [8], Chapter 15). Let $x_1 < x_2 < \cdots$ be the positive, real zeros of J_{-H} and let $y_1 < y_2 < \cdots$ be the positive, real zeros of J_{1-H}.

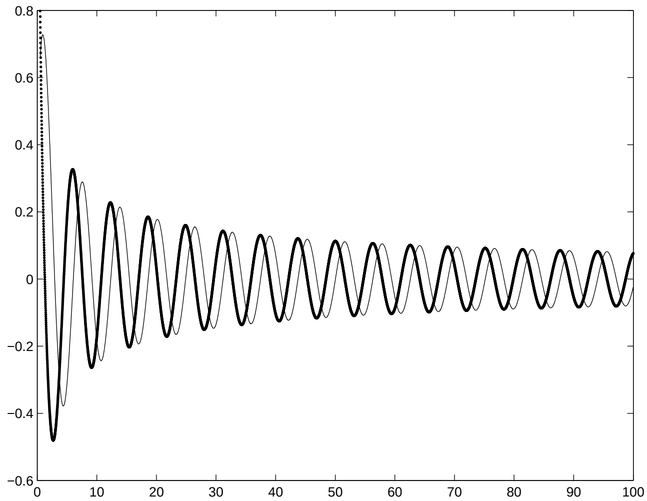


Figure 1: Bessel functions: J_{-H} (with '·'), J_{1-H} (with '-'), where $H = 0.65$

Let $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ be two independent sequences of independent Gaussian random variables such that for each $n \in \mathbb{N}$ we have

$$E(X_n) = E(Y_n) = 0$$

and

$$\mathrm{Var}X_n = \frac{2c_H^2(t)}{x_n^{2H(t)} J^2_{1-H(t)}(x_n)}, \quad \mathrm{Var}Y_n = \frac{2c_H^2}{y_n^{2H(t)} J^2_{-H(t)}(y_n)},$$

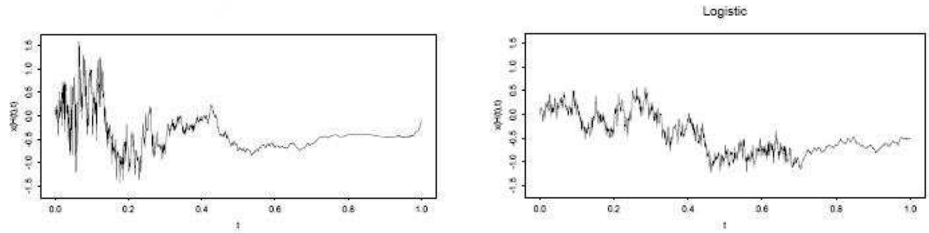
where

$$c_H^2 = \frac{\sin(\pi H(t))}{\pi}\Gamma(1+2H).$$

In [2] it is proved that the random process $B = \Big(B(t)\Big)_{t\in[0,1]}$ given by

$$B(t) = \sum_{n=1}^{\infty} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1]$$

is well defined and both series converge absolutely and uniformly in $t \in [0,1]$. The process B is a fractional Brownian motion with Hurst index H. Similarly if we use the Hurst index $H(t)$ we obtain an approximation for multifractional Brownian motion.



a) linear b) logistic

Figure 2: Approximation B_N of multifractional Brownian motion

For each $N \in \mathbb{N}$ we define the process

$$B_N(t) = \sum_{n=1}^{N} \frac{\sin(x_n t)}{x_n} X_n + \sum_{n=1}^{N} \frac{1-\cos(y_n t)}{y_n} Y_n, \quad t \in [0,1], \tag{2.2}$$

then using the above mentioned result from [2] we have

$$P(\lim_{N\to\infty} \sup_{t\in[0,1]} |B(t) - B_N(t)| = 0) = 1. \tag{2.3}$$

In the sequel we need the following result:

Theorem 2.1. *For all $N \in \mathbb{N}$ the approximating processes $(B_N(t))_{t\in[0,1]}$ are with probability 1 Lipschitz continuous.*

Proof. Let $N \in \mathbb{N}$ be fixed. We write

$$|B_N(t) - B_N(s)| \leqslant \sum_{n=1}^{N} \left| \frac{\sin(x_n t) - \sin(x_n s)}{x_n} X_n \right| + \sum_{n=1}^{N} \left| \frac{\cos(x_n s) - \cos(x_n t)}{x_n} Y_n \right|$$

But the functions sin and cos are Lipschitz continuous, therefore

$$|B_N(t) - B_N(s)| \leqslant |t-s| \sum_{n=1}^{N} \Big(|X_n| + |Y_n| \Big) = C_N |t-s|, \quad \text{for all } s,t \in [0,1],$$

where $C_N = \sum_{n=1}^{N} \Big(|X_n| + |Y_n| \Big) < \infty$ is a random variable. $\square$

3. Fractional integrals and derivatives

Let $a, b \in \mathbb{R}$, $a < b$ and $f, g : \mathbb{R} \to \mathbb{R}$. We use notions and results about fractional calculus, from [7]:

$$f(a+) := \lim_{\delta \searrow 0} f(a+\delta), \quad f(b-) := \lim_{\delta \searrow 0} f(b-\delta),$$

$$f_{a+}(x) = \mathbb{I}_{(a,b)}(f(x) - f(a+)), \quad g_{b-}(x) = \mathbb{I}_{(a,b)}(g(x) - g(b-)).$$

Note that for $\alpha > 0$ we have $(-1)^\alpha = e^{i\pi\alpha}$.

For $f \in L_1$ and $\alpha > 0$ the *left-sided fractional Riemann-Liouville integral of f of order α* on (a, b) is given for a.e. x by

$$I^\alpha_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) dy$$

and the *right-sided fractional Rieman-Liouville integral of f of order α* on (a, b) is given for a.e. x by

$$I^\alpha_{b-} f(x) = \frac{(-1)^{-\alpha}}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha-1} f(y) dy.$$

Fractional differentiation may be introduced as an inverse operation to fractional integration. For our purposes it is sufficient to work with a class of functions where this inversion is well-determined and the Riemann-Liouville derivatives agree with the fractional derivatives in the sense of Weyl and Marchaud.

For $p > 1$ let $I^\alpha_{a+}(L_p(a, b))$ be the class of functions f which have the representation $f = I^\alpha_{a+}\Phi$, where $\Phi \in L_p(a, b)$. If $0 < \alpha < 1$, then the function Φ in this representation agrees a.e. with the *left-sided Riemann-Liouville derivative of f of order α*

$$D^\alpha_{a+} f(x) := \mathbb{I}_{(a,b)}(x) \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x \frac{f(y)}{(x-y)^\alpha} dy,$$

where the corresponding Weyl representation is

$$D^\alpha_{a+} f(x) := \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(x)}{(x-a)^\alpha} + \alpha \int_a^x \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \right) \mathbb{I}_{(a,b)}(x).$$

It is known that $f \in I^\alpha_{a+}(L_p(a, b))$ if and only if $f \in L_p(a, b)$ and the integral

$$\mathcal{I}_\varepsilon(x) = \int_a^{x-\varepsilon} \frac{f(x) - f(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (a, b)$$

converges in $L_p(a, b)$ as $\varepsilon \searrow 0$.

For $p > 1$ let $I^\alpha_{b-}(L_p(a, b))$ be the class of functions f which have the representation $g = I^\alpha_{b-}\Phi$, where $\Phi \in L_p(a, b)$. If $0 < \alpha < 1$, then the function Φ in this representation agrees a.e. with the *right-sided Riemann-Liouville derivative of$g$ of order α* (given in the Weyl representation)

$$D^\alpha_{b-} g(x) := \mathbb{I}_{(a,b)}(x) \frac{(-1)^{1+\alpha}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b \frac{g(y)}{(y-x)^\alpha} dy,$$

where the corresponding Weyl representation is

$$D^{\alpha}_{b-}g(x) := \frac{(-1)^{\alpha}}{\Gamma(1-\alpha)}\left(\frac{g(x)}{(b-x)^{\alpha}} + \alpha\int\limits_{x}^{b}\frac{g(x)-g(y)}{(y-x)^{\alpha+1}}dy\right)\mathbb{I}_{(a,b)}(x).$$

It is known that $g \in I^{\alpha}_{b-}(L_p(a,b))$ if and only if $g \in L_p(a,b)$ and the integral

$$\mathcal{J}_{\varepsilon}(x) = \int\limits_{x+\varepsilon}^{b}\frac{f(x)-f(y)}{(y-x)^{\alpha+1}}dy \text{ for } x \in (a,b)$$

converges in $L_p(a,b)$ as $\varepsilon \searrow 0$. Recall that by construction we have for $f \in I^{\alpha}_{a+}(L_p(a,b))$ and $g \in I^{\alpha}_{b-}(L_p(a,b))$

$$I^{\alpha}_{a+}(D^{\alpha}_{a+}f) = f, \quad I^{\alpha}_{b-}(D^{\alpha}_{b-}g) = g \tag{3.1}$$

and

$$D^{\alpha}_{a+}(I^{\alpha}_{a+}f) = f, \quad D^{\alpha}_{b-}(I^{\alpha}_{b-}g) = g. \tag{3.2}$$

In [9] is defined the *fractional integral* of a function f with respect to g as follows

$$\begin{aligned}\int\limits_{a}^{b} f(x)dg(x) \quad &= \quad (-1)^{\alpha}\int\limits_{a}^{b} D^{\alpha}_{a+}f_{a+}(x)D^{1-\alpha}_{b-}g_{b-}(x)dx \\ &\quad +f(a+)(g(b-)-g(a+))\end{aligned} \tag{3.3}$$

if $f_{a+} \in I^{\alpha}_{a+}(L_p(a,b)), g_{b-} \in I^{1-\alpha}_{b-}(L_q(a,b))$ for some $\frac{1}{p}+\frac{1}{q} \leqslant 1$ and $0 \leqslant \alpha \leqslant 1$.

In our investigations we will take $p=q=2$. If $0 \leqslant \alpha < \frac{1}{2}$, then the integral in (3.3) can be written as

$$\int\limits_{a}^{b} f(x)dg(x) = (-1)^{\alpha}\int\limits_{a}^{b} D^{\alpha}_{a+}f(x)D^{1-\alpha}_{b-}g_{b-}(x)dx \tag{3.4}$$

if $f \in I^{\alpha}_{a+}(L_2(a,b))$, $f(a+)$ exists, $g_{b-} \in I^{1-\alpha}_{b-}(L_2(a,b))$.

4. The stochastic integral

Without loss of generality we consider $0 < T \leqslant 1$, because for arbitrary $T > 0$ we can rescale the time variable using the $H(t)$-self similar property of the multifractional Brownian motion meaning that $\Big(B(ct)\Big)_{t\geqslant 0}$ and $\Big(c^{H(t)}B(t)\Big)_{t\geqslant 0}$ are equal in distribution for every $c > 0$.

We will define the $\int\limits_0^T G(u)dB(u)$ Ito integral instead of $\int\limits_0^t G(u)dB(u)$ and use

$$\int\limits_0^t G(u)dB(u) = \int\limits_0^T \mathbb{I}_{[0,t]}(u)G(u)dB(u) \text{ for } t \in [0,T]$$

(by Theorem 2.5 in [9]).

We consider $\alpha > 1 - H$. It follows by (3.4) that

$$\int\limits_0^T G(u)dB(u) = (-1)^\alpha \int\limits_0^T D_{0+}^\alpha G(u) D_{T-}^{1-\alpha} B_{T-}(u)du \tag{4.1}$$

for $G \in I_{0+}^\alpha(L_2(0,T))$, where $G(0+)$ exists and $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$.

The condition $G \in I_{0+}^\alpha(L_2(0,T))$ (with probability 1) means that $G \in L_2(0,T)$ and

$$\mathcal{I}_\varepsilon(x) = \int\limits_0^{x-\varepsilon} \frac{G(x) - G(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (0,T)$$

converges in $L^2(0,T)$ as $\varepsilon \searrow 0$, i.e. there exists in $L_2(0,T)$ the indefinite integral

$$\mathcal{I}(x) = \int\limits_0^{x} \frac{G(x) - G(y)}{(x-y)^{\alpha+1}} dy \text{ for } x \in (0,T)$$

such that

$$\lim_{\varepsilon \searrow 0} \int\limits_0^T \Big(\mathcal{I}_\varepsilon(x) - \mathcal{I}(x)\Big)^2 dx = 0.$$

The condition $B_{T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$ means $B_{T-} \in L_2(0,T)$ and

$$\mathcal{J}_\varepsilon(x) = \int\limits_{x+\varepsilon}^{T} \frac{B(x) - B(y)}{(y-x)^{2-\alpha}} dy \text{ for } x \in (0,T)$$

converges in $L_2(0,T)$ as $\varepsilon \searrow 0$, i.e. there exists in $L_2(0,T)$ the integral

$$\mathcal{J}(x) = \int\limits_{x}^{T} \frac{B(x) - B(y)}{(y-x)^{2-\alpha}} dy \text{ for } x \in (0,T)$$

such that

$$\lim_{\varepsilon \searrow 0} \int\limits_0^T \Big(\mathcal{J}_\varepsilon(x) - \mathcal{J}(x)\Big)^2 dx = 0.$$

This condition for B is fulfilled for $\alpha > 1-H$, since the fractional Brownian motion B is a.s. Hölder continuous with exponent $\gamma \in (0,H)$ (see [1]), i.e.

$$\mathbb{P}\left(\omega \in \Omega : \sup_{\substack{0<t-u<h(\omega)\\ t,u\in[0,T]}} \frac{|B(t)-B(u)|}{|t-u|^{\gamma}} \leqslant \delta\right) = 1,$$

where h is an a.s. positive random variable and $\delta > 0$ is an appropriate constant.

We will use (3.4) for the integrals with respect the approximating processes $\Big(B_N(t)\Big)_{t\in[0,T]}$. Observe that $B_{N,T-} \in I_{T-}^{1-\alpha}(L_2(0,T))$, which follows from the Lipschitz continuity property in Theorem 2.1. We have

$$\int_0^T G(u)dB_N(u) = (-1)^{\alpha} \int_0^T D_{0+}^{\alpha} G(u) D_{T-}^{1-\alpha} B_{N,T-}(u)du \tag{4.2}$$

for $G \in I_{0+}^{\alpha}(L_2(0,T))$, where $G(0+)$ exists.

Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a cádlág process. Its *generalized quadratic variation process* $\Big([Z](t)\Big)_{t\in[0,T]}$ is defined as

$$\begin{aligned}[Z](t) \quad &= \quad \lim_{\varepsilon\searrow 0} \varepsilon \int_0^1 \int_0^t \frac{1}{u}(Z_{t-}(s+u) - Z_{t-}(s))^2 ds du \\ &\quad + (Z(t) - Z(t-))^2,\end{aligned}$$

if the limit exists uniformly in probability (see [6], also in [10] Section 5).

In particular, if B is a fractional Brownian motion with Hurst index $H \in (\frac{1}{2},1)$ and B_N $(N\in\mathbb{N})$ is an approximation of B as given in (2.2), it is easy to verify that

$$[B](t) = 0 \quad \text{and} \quad [B_N](t) = 0 \quad \text{for each } \ t\in[0,T], \tag{4.3}$$

because B is locally Hölder continuous and B^N is Lipschitz continuous. The *Ito formula* for change of variable in fractal-type integrals is given in the next theorem.

Theorem 4.1 ([10], Theorem 5.8). *Let $\Big(Z(t)\Big)_{t\in[0,T]}$ be a continuous process with generalized quadratic variation $[Z]$. Let $Q : \mathbb{R}\times[0,T] \to \mathbb{R}$ be a random function such that with probability 1 we have $Q \in \mathcal{C}^1(\mathbb{R}\times[0,T])$ and $\frac{\partial^2 Q}{\partial x^2} \in \mathcal{C}(\mathbb{R}\times[0,T])$. Then, for $t_0, t \in [0,T]$ we have*

$$\begin{aligned}Q(Z(t),t) - Q(Z(t_0),t_0) \quad &= \quad \int_{t_0}^t \frac{\partial Q}{\partial x}(Z(s),s)dZ(s) + \int_{t_0}^t \frac{\partial Q}{\partial t}(Z(s),s)ds \\ &\quad + \int_{t_0}^t \frac{\partial^2 Q}{\partial^2 x}(Z(s),s)d[Z]s.\end{aligned}$$

Let $1 - H(t) < \alpha < \frac{1}{2}$ and let $G \in I^{\alpha}_{0+}(L_2(0,T))$ such that $G(0+)$ exists. We define the processes

$$Z(t) = \int_0^t G(s)dB(s) \text{ and } Z_N(t) = \int_0^t G(s)dB_N(s), \quad t \in (0,T].$$

Then by Theorem 5.6, p. 167 in [10] it follows that

$$[Z](t) = 0 \text{ and } [Z_N](t) = 0.$$

Using Theorem 4.1, it follows that, if $Q : \mathbb{R} \times [0,T] \to \mathbb{R}$ is a random function such that with probability 1 we have $Q \in \mathcal{C}^1(\mathbb{R} \times [0,T])$ and $\frac{\partial^2 Q}{\partial x^2} \in \mathcal{C}(\mathbb{R} \times [0,T])$, then for $t_0, t \in [0,T]$ we have

$$\begin{aligned} Q(Z(t),t) - Q(Z(t_0),t_0) &= \int_{t_0}^t \frac{\partial Q}{\partial x}(Z(s),s)G(s)dB(s) \\ &\quad + \int_{t_0}^t \frac{\partial Q}{\partial t}(Z(s),s)ds \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} Q(Z_N(t),t) - Q(Z_N(t_0),t_0) &= \int_{t_0}^t \frac{\partial Q}{\partial x}(Z_N(s),s)G(s)dB_N(s) \\ &\quad + \int_{t_0}^t \frac{\partial Q}{\partial t}(Z_N(s),s)ds. \end{aligned} \tag{4.5}$$

5. Some results

We prove the existence of the local solution of a deterministic equation with locally Lipschitz function (in the version we need in our paper). We adapt the ideas from the proof of Theorem 1.4 in [5]. We give the proof here in order to make the proof of Theorem 5.2 more understandable.

In what follows $\|\cdot\|$ denotes the norm in $\mathbb{R}^n$.

Theorem 5.1. *Let $A : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^n$ be such that for each $u \in \mathbb{R}^n$ the function $A(u,\cdot)$ is continuous and for any $c, T > 0$ we have*

$$\|A(x,t) - A(y,t)\| \leqslant L(c,T)\|x - y\|$$

for all $x, y \in \mathbb{R}^n$ *with* $\|x\| \leqslant c, \|y\| \leqslant c$ *and* $t \in [0,T]$*, where* $L(c,T) > 0$ *is the locally Lipschitz constant. We consider the equation*

$$U(t) = U_0 + \int_{t_0}^{t} A(U(s), s) ds, \tag{5.1}$$

where $U_0 \in \mathbb{R}^n$ *and* $t_0 > 0$ *fixed. Then equation (5.1) has a local solution, i.e. there exists a maximal interval* $(t_1, t_2) \in [0, \infty)$ *containing* t_0 *and a function* $U : \mathbb{R}^n \times (t_1, t_2) \to \mathbb{R}^n$ *such that (5.1) is satisfied for each* $t \in (t_1, t_2)$*.*

Proof. For any $\tau > 0$ let $M(\tau) = \max_{t \in [0, \tau+1]} \|A(0,t)\|$. We consider

$$\delta = \min\left\{1, \frac{\|U_0\|}{2\|U_0\| L(2\|U_0\|, t_0+1) + M(t_0)}, t_0\right\}.$$

We define the mapping $\mathcal{A} : C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n) \to C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$

$$(\mathcal{A}U)(t) := U_0 + \int_{t_0}^{t} A(U(s), s) ds, \quad t \in [t_0 - \delta, t_0 + \delta].$$

We prove that $\mathcal{A}$ maps the ball $\mathcal{B}(0, R)$ of radius $R = 2\|U_0\|$ centered at 0 of the space $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$ into itself. For $U \in \mathcal{B}(0,R)$ and for each $t \in [t_0 - \delta, t_0 + \delta]$ we have the following estimates

$$\begin{aligned} \|\mathcal{A}(U)(t)\| &\leqslant \|U_0\| + \left| \int_{t_0}^{t} \|A(U(s), s) - A(0,s)\| + \|A(0,s)\|\Big) ds \right| \\ &\leqslant \|U_0\| + (L(R, t_0+1)R + M(t_0))|t - t_0| \leqslant 2\|U_0\| = R. \end{aligned}$$

Therefore, $\mathcal{A}U \in \mathcal{B}(0,R)$. It is easy to verify that for each $U, V \in \mathcal{B}(0,R)$ and each $t \in [t_0 - \delta, t_0 + \delta]$ we have

$$\|\mathcal{A}(U)(t) - \mathcal{A}(V)(t)\| \leqslant L(R, t_0+1)|t - t_0| \sup_{t \in [t_0 - \delta, t_0 + \delta]} \|U(t) - V(t)\|.$$

For each $N \in \mathbb{N}$ we denote

$$\mathcal{A}^N = \underbrace{\mathcal{A} \circ \cdots \circ \mathcal{A}}_{N \text{ times}}.$$

From the definition of $\mathcal{A}$ it then follows for each $N \in \mathbb{N}$ and each $t \in [t_0 - \delta, t_0 + \delta]$ that

$$\|\mathcal{A}^N(U)(t) - \mathcal{A}^N(V)(t)\| \leqslant \frac{\Big(L(R, t_0+1)|t - t_0|\Big)^N}{N!} \sup_{t \in [t_0 - \delta, t_0 + \delta]} \|U(t) - V(t)\|.$$

Hence

$$\sup_{t\in[t_0-\delta,t_0+\delta]} \|\mathcal{A}^N(U)(t)-\mathcal{A}^N(V)(t)\| \leqslant \frac{\Big(L(R,t_0+1)\delta\Big)^N}{N!} \sup_{t\in[t_0-\delta,t_0+\delta]} \|U(t)-V(t)\|.$$

For N large enough we have $\dfrac{\Big(L(R,t_0+1)\delta\Big)^N}{N!} < 1$. By a well known extension of the contraction principle it follows that $\mathcal{A}$ has a unique fixed point in $\mathcal{B}(0,R)$.

We have proved that there exists a solution U defined on the interval $[t_0-\delta, t_0+\delta]$ satisfying (5.1). This solution can be extended to the interval $[t_0-\delta^*, t_0+\delta^*]$ $(\delta^* > \delta)$, where on $[t_0-\delta, t_0+\delta]$ we have the above solution U and for $t \geqslant t_0+\delta$ we use the above method to find a local solution for

$$U(t) = U(t_0+\delta) + \int_{t_0+\delta}^{t} A(U(s),s)ds,$$

and also for $t \leqslant t_0-\delta$ we use the above method to find a local solution for

$$U(t) = U(t_0-\delta) + \int_{t_0-\delta}^{t} A(U(s),s)ds.$$

Moreover, δ^* depends only on $\delta, \|U(t_0+\delta)\|, \|U(t_0-\delta)\|, M(t_0+\delta), M(t_0-\delta)$. Hence, there exists a maximal interval (t_1,t_2) containing t_0 for the existence of the local solution U. □

Theorem 5.2. *Let $A : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}^n$ be such that for each $(x,u) \in \mathbb{R}^{n+1}$ the function $A(x,u,\cdot)$ is continuous and we have*

$$\|A(x,u,t) - A(y,v,t)\| \leqslant L(c)(\|x-y\| + |u-v|)$$

for all $x, y \in \mathbb{R}^n$ with $\|x\| \leqslant c, \|y\| \leqslant c, |u| \leqslant c, |v| \leqslant c$ and each $t \in [0,T]$, where $L(c) > 0$ is the locally Lipschitz constant. Let $U_0 \in \mathbb{R}^n$ and $t_0 \in (0,T]$ fixed. Assume that $(v_N)_{N\in\mathbb{N}}$ is a sequence from $C[0,T]$ which converges uniformly to $v \in C[0,T]$, i.e.

$$\lim_{N\to\infty} \sup_{t\in[0,T]} |v_N(t) - v(t)| = 0.$$

We consider the equations

$$U_N(t) = U_0 + \int_{t_0}^{t} A(U_N(s), v_N(s), s)ds, \quad N \in \mathbb{N} \tag{5.2}$$

and

$$U(t) = U_0 + \int_{t_0}^{t} A(U(s), v(s), s)ds. \tag{5.3}$$

The equations (5.2) *and* (5.3) *have local solutions, i.e. there exists a maximal interval* $(t_1, t_2) \subset [0,T]$ *(which does not depend on* N*) containing* t_0 *and functions* $U_N, U : \mathbb{R}^n \times (t_1, t_2) \to \mathbb{R}^n$ *such that* (5.2) *and* (5.3) *are satisfied for each* $t \in (t_1, t_2)$*. Moreover,*

$$\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|U_N(t) - U(t)\| = 0.$$

Proof. For any $\tau > 0$ let $M = \max_{t\in[0,T]} \|A(0,0,t)\|$. Since $(v_N)_{N\in\mathbb{N}}$ converges uniformly to v in $C[0,T]$, it follows that there exists $m > 0$ such that

$$\sup_{t\in[0,T]} |v_N(t)| + \sup_{t\in[0,T]} |v(t)| \leqslant m \qquad \text{for each } N \in \mathbb{N}.$$

We consider

$$\delta = \min\left\{1, \frac{m}{(\|U_0\| + 2m)L(\|U_0\| + m) + M}, t_0, T - t_0\right\}.$$

We define the mapping $\mathcal{F}_N : C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n) \to C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$

$$(\mathcal{F}_N Y)(t) := U_0 + \int_{t_0}^{t} A(Y(s), v_N(s), s)ds, \quad t \in [t_0 - \delta, t_0 + \delta].$$

We prove that $\mathcal{F}_N$ maps the ball $\mathcal{B}(0,R)$ of radius $R = \|U_0\| + m$ centered at 0 of the space $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^n)$ into itself. For $Y \in \mathcal{B}(0,R)$ and for each $t \in [t_0 - \delta, t_0 + \delta]$ we have the following estimates

$$\begin{aligned}
\|\mathcal{F}_N(Y)(t)\| &\leqslant \|U_0\| + \left|\int_{t_0}^{t} \|A(Y(s), v_N(s), s) - A(0,0,s)\| + \|A(0,0,s)\|\Big)ds\right| \\
&\leqslant \|U_0\| + (L(R)(R+m) + M)|t - t_0| \leqslant \|U_0\| + m = R.
\end{aligned}$$

Therefore, $\mathcal{F}_N Y \in \mathcal{B}(0,R)$. It is easy to verify that for each $Y, Z \in \mathcal{B}(0,R)$ and each $t \in [t_0 - \delta, t_0 + \delta]$ we have

$$\|\mathcal{F}_N(Y)(t) - \mathcal{F}_N(Z)(t)\| \leqslant L(R)|t - t_0| \sup_{t\in[t_0-\delta,t_0+\delta]} \|Y(t) - Z(t)\|.$$

Using the contraction principle exactly as in the proof of Theorem 5.1, it follows that $\mathcal{F}_N$ has a unique fixed point in $\mathcal{B}(0,R)$, which is defined on $[t_0 - \delta, t_0 + \delta]$. This fixed point is the local solution U_N of (5.2). We observe that this interval of existence of the local solution U_N does not depend on N, and $U_N \in \mathcal{B}(0,R)$ for each $N \in \mathbb{N}$. Exactly in the same way we can prove that on the same interval $[t_0-\delta, t_0+\delta]$ there exists a solution $U \in \mathcal{B}(0,R)$ satisfying (5.3). Let $(t_1, t_2) \subset (0,T]$ be the maximal interval (which does not depend on N) containing t_0 such that (5.2) and (5.3) are satisfied for each $t \in (t_1, t_2)$ and there exists $R > 0$ (independent of N) such that $U_N, U \in \mathcal{B}(0, R)$. Then for large N we have

$$\begin{aligned} \|U_N(t) - U(t)\| &\leqslant \left| \int_{t_0}^{t} \|A(U_N(s), v_N(s), s) - A(U(s), v(s), s)\| ds \right| \\ &\leqslant \left| \int_{t_0}^{t} L(R)(\|U_N(s) - U(s)\| + \|v_N(s) - v(s)\|) ds \right|. \end{aligned}$$

By the Gronwall lemma we get

$$\sup_{t \in (t_1, t_2)} \|U_N(t) - U(t)\| \leqslant \sup_{t \in (t_1, t_2)} \|v_N(t) - v(t)\| e^{L(R)(t_2 - t_1)}$$

Therefore,

$$\lim_{N \to \infty} \sup_{t \in (t_1, t_2)} \|U_N(t) - U(t)\| = 0.$$

□

6. Stochastic differential equations driven by multifractional Brownian motion

Let $\left(B(t)\right)_{t \geqslant 0}$ be a multifractional Brownian motion with Hurst parameter $H(t)$ such that $H > \frac{1}{2}$. We investigate stochastic differential equations of the form

$$\begin{aligned} dX(t) &= F(X(t), t)dt + G(X(t), t)dB(t), \\ X(t_0) &= X_0, \end{aligned} \tag{6.1}$$

where X_0 is a random vector in $\mathbb{R}^n$ and the random functions F and G satisfy with probability 1 the following conditions:

(C1) $F \in C(\mathbb{R}^n \times [0, T], \mathbb{R}^n), G \in C^1(\mathbb{R}^n \times [0, T], \mathbb{R}^n)$;

(C2) for each $t \in [0, T]$ the functions $F(\cdot, t), \dfrac{\partial G(\cdot, t)}{\partial x^i}, \dfrac{\partial G(\cdot, t)}{\partial t}$ are locally Lipschitz for each $i \in \{1, \ldots, n\}$.

We consider the pathwise auxiliary partial differential equation on $\mathbb{R}^n \times \mathbb{R} \times [0, T]$

$$\begin{aligned} \frac{\partial K}{\partial z}(y, z, t) &= G(K(y, z, t), t), \\ K(Y_0, Z_0, t_0) &= X_0, \end{aligned} \tag{6.2}$$

where Y_0 is an arbitrary random vector in $\mathbb{R}^n$ and Z_0 an arbitrary random variable in $\mathbb{R}$. From the theory of differential equations it follows that with probability

1 there exists a local solution $K \in C^1(\mathbb{R}^n \times [0,T], \mathbb{R}^n)$ in a neighbourhood V of (Y_0, Z_0, t_0) with partial derivatives being Lipschitz in the variable y and

$$\det\left(\frac{K^i}{\partial y^j}(y,z,t)\right)_{1\leqslant i,j\leqslant n} \neq 0.$$

We have for $(x,y,t) \in V$

$$\frac{\partial^2 K}{\partial z^2}(y,z,t) = \sum_{j=1}^{n} \frac{\partial G}{\partial x^j}(K(y,z,t),t)G^j(K(y,z,t),t).$$

We also consider the pathwise differential equation (in matrix representation) on $[0,T]$

$$\begin{aligned} dY(t) &= \left(\frac{K}{\partial y}(Y(t),B(t),t)\right)^{-1}\Big[F(K(Y(t),B(t),t),t) \\ &\qquad -\frac{\partial K}{\partial t}(Y(t),B(t),t)\Big]dt \\ Y(t_0) &= Y_0, \end{aligned}$$

which has a unique local solution on a maximal interval $(t_0^1, t_0^2) \subseteq [0,T]$ with $t_0 \in (t_0^1, t_0^2)$ (see Theorem 5.1).

Applying the Ito formula, see Theorem 4.1 and relation (4.4), to the random function $Q(z,t) = K(Y(t),z,t)$ (successively for $K^1, \dots, K^n$) and the fractional Brownian motion B we obtain

$$\begin{aligned} &K(Y(t),B(t),t) - K(Y(t_0),B(t_0),t_0) \\ &= \sum_{j=1}^{n}\int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s) + \int_{t_0}^{t} \frac{\partial K}{\partial z}(Y(s),B(s),s)dB(s) \\ &\qquad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y(s),B(s),s)ds \\ &= \sum_{j=1}^{n}\int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y(s),B(s),s)dY^j(s) + \int_{t_0}^{t} G(K(Y(s),B(s),s),s)dB(s) \\ &\qquad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y(s),B(s),s)ds \\ &= \int_{t_0}^{t} F(K(Y(s),B(s),s),s)ds + \int_{t_0}^{t} G(K(Y(s),B(s),s),s)dB(s). \end{aligned}$$

Therefore,

$$X(t) := K(Y(t),B(t),t)$$

satisfies

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s), s)ds + \int_{t_0}^{t} G(X(s), s)dB(s).$$

Instead of the process $\Big(B(t)\Big)_{t\in[0,1]}$ we consider its approximations $\Big(B_N(t)\Big)_{t\in[0,1]}$, $N \in \mathbb{N}$ given in (2.2). For each $N \in \mathbb{N}$ we consider the pathwise differential equation (in matrix representation)

$$\begin{aligned} dY_N(t) &= \left(\frac{\partial K}{\partial y}(Y_N(t), B_N(t), t)\right)^{-1} \Big[F(K(Y_N(t), B_N(t), t), t) \\ &\qquad -\frac{\partial K}{\partial t}(Y_N(t), B_N(t), t)\Big] dt \\ Y_N(t_0) &= Y_0, \end{aligned}$$

which has a unique local solution Y_N on a maximal interval $(t^1, t^2) \subset (t_1^0, t_2^0)$ of existence which contains t_0 (see Theorem 5.2). Applying the Ito formula, see Theorem 4.1 and (4.5), to the random function $Q(z, t) = K(Y_N(t), z, t)$ (successively for $K^1, \ldots, K^n$) we obtain

$$\begin{aligned} & K(Y_N(t), B_N(t), t) - K(Y_N(t_0), B_N(t_0), t_0) \\ &= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y_N(s), B_N(s), s)dY_N^j(s) + \int_{t_0}^{t} \frac{\partial K}{\partial z}(Y_N(s), B_N(s), s)dB_N(s) \\ &\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y_N(s), B_N(s), s)ds \end{aligned}$$

$$\begin{aligned} &= \sum_{j=1}^{n} \int_{t_0}^{t} \frac{\partial K}{\partial y^j}(Y_N(s), B_N(s), s)dY_N^j(s) + \int_{t_0}^{t} G(K(Y_N(s), B_N(s), s), s)dB_N(s) \\ &\quad + \int_{t_0}^{t} \frac{\partial K}{\partial t}(Y_N(s), B_N(s), s)ds \\ &= \int_{t_0}^{t} F(K(Y(s), B(s), s), s)ds + \int_{t_0}^{t} G(K(Y_N(s), B_N(s), s), s)dB_N(s). \end{aligned}$$

Therefore,

$$X_N(t) := K(Y_N(t), B_N(t), t)$$

satisfies

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s), s)ds + \int_{t_0}^{t} G(X_N(s), s)dB_N(s), \quad t \in (t_1, t_2).$$

By Theorem 5.2 it follows that we have the following pathwise property

$$\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|Y_N(t) - Y(t)\| = 0.$$

Then by the continuity properties of K and by (2.2) it follows that for a.e. $\omega \in \Omega$ we have

$$\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|X_N(t) - X(t)\| = 0.$$

By this we have proved the main result of our paper:

Theorem 6.1. *Let B be a multifractional Brownian motion approximated through the processes B_N given in* (2.2) *and* (2.3). *Let $F, G : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ be random functions satisfying with probability* 1 *the conditions* (C1) *and* (C2). *Let $t_0 \in (0,T]$ be fixed. Then, each of the stochastic equations*

$$X(t) = X_0 + \int_{t_0}^{t} F(X(s), s)ds + \int_{t_0}^{t} G(X(s), s)dB(s),$$

$$X_N(t) = X_0 + \int_{t_0}^{t} F(X_N(s), s)ds + \int_{t_0}^{t} G(X_N(s), s)dB_N(s), \quad N \in \mathbb{N}$$

admit almost surely a unique local solution on a common interval (t_1, t_2) (which is independent of N). Moreover, we have the following approximation result

$$P(\lim_{N\to\infty} \sup_{t\in(t_1,t_2)} \|X_N(t) - X(t)\| = 0) = 1.$$

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