Proceedings of the 7th International Conference on Applied Informatics
Eger, Hungary, January 28–31, 2007. Vol. 1. pp. 185–192.

Two- and three-dimensional tiling on the base of higher-dimensional cube mosaics

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Abstract

Lifting the vertices of a k sided regular polygon from it is plane, perpendicularly by the same height, and joining with the centre of the polygon, we get the k edges of the hypercube modelled in the three-dimensional space. From these the 3-models or its surface polyhedra can be generated as well in different procedures. Combining $2 < j < k$ edges, 3-models of j-cubes, as parts of the k-cube, are easy to build.

The space-filling arrangement of these models can further be dissected and reordered by inner lower-dimensional 3-models and Boole-operations.

In the models for any k each vertex lies in planes parallel to the base plane of construction and a plane tiling appears in these intersections. These tiling can further be dissected and reordered.

Some k-cube 3-models can also create space-filling polyhedra. The cases $k > 9$ are currently under research.

Remark. The topic requires colored figures with transparent solids. This proceeding contains only some figures which can be understood more or less in black and white printing. The lecture can be downloaded with many figures as a powerpoint file from: `http://icai.voros.pmmf.hu`

Lifting the vertices of a k sided regular polygon from it is plane, perpendicularly by the same height, and joining with the centre of the polygon, we get the k edges of the hypercube (k-cube) modelled in the three-dimensional space (3-model). From these the 3-models or their polyhedral surface (Figure 1) can be generated as well in different procedures [4, 5, 6]. Each polyhedron from these will be a so called zonotope/zonohedra [3], i.e. a “translational sum” (Minkowski-sum) of some segments [4]. Combining $2 < j < k$ edges, 3-models of j-cubes, as parts of the k-cube, are easy to build.

The 2-dimensional ortogonal projection of these 3-models indicates the idea how to construct space-filling with this model. However our 3-model of the 6-cube for example does not fill the space. The projected grid of the 3-cube joins our grid above and the cube fills the space well known. The edges of the cube can be selected from the conveniant lifted edges of the 6-cube’s 3-model. With the selected four edges of the grid we can build the 3-model of the 4-cube. The shell of this is a rhombic dodekahedron which fills the space but this arrangement has not any rotational symmetry without additional assumptions. We can however replace a cube in the hole of the rotational-symmetrically arranged rhombic dodekahedra and continue the filling in a sixfold polar array with a rhombic triacontahedron which contains our 3-model of the 6-cube (Figure 2). It can be seen, that we can fill the space with these solids. The basic stones are to cut from a honeycomb by symmetry-planes.

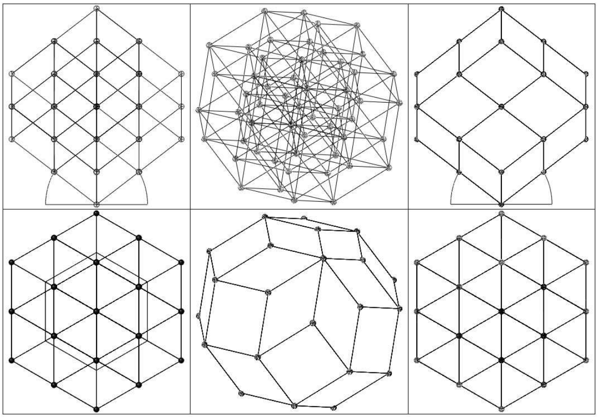


Figure 1

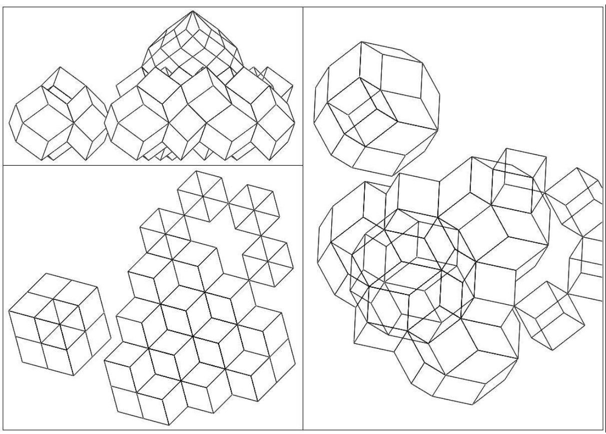


Figure 2

An other possibility is to rearrange our space-filling, assembling the 3-models of the k- and j-cubes from lower-dimensional cube 3-models. From the given edges we can combine the 3-models of $2 < j < k$ cubes. In case $k = 6$: 4 of the 3-cubes, 3 of the 4-cubes and 1 of the 5-cubes. Their additions (Figure 3) can replace the 3-models of the above k- and j-cubes in our mosaic.

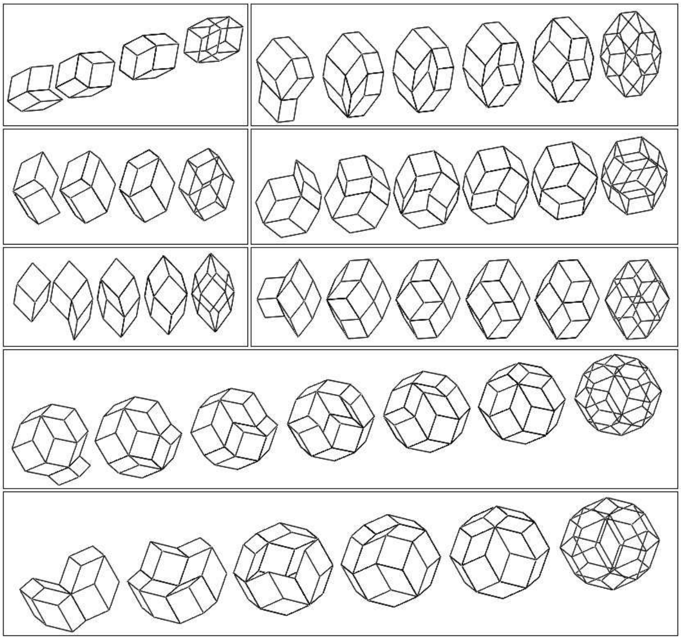
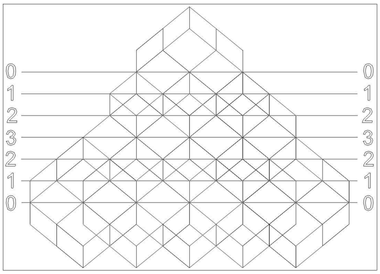


Figure 3

Interpreting the starting construction of the k-cube 3-model as a sequence of dispositions, the increasing dimensional inner $2 < j < k$ cube 3-models can "easily" be separated. The edges of the $0, 1, \ldots, k$ cube model-sequence are parallel to the k-segment chain approaching a starting helix, and the disposition vectors are joining each other along this segment chain. The model $0, 1, \ldots, k-1$ parts can also be interpreted as intersections of two full models so that the equal dimensional parts are positioned around the main diagonal of a full model, symmetrically to its centre point. More on this full model (3-model of the n-cube) can be read in [5, 6, 8], and on periodic and aperiodic tilings, based on d-dimensional crystallographic space groups, you find references in [1].

As it follows from our construction, the vertices lie in planes parallel to the basic plane of the construction, therefore a plane-tiling appears on these horizontal intersections of our space-filling solid-mosaics based on the 3-model of the k-cube (Figure 4). This has rotational symmet-ries but for example in case $k = 6$ the diagonal intersections can be identical with the longitu-dinal and cross-intersections (Figure 5).



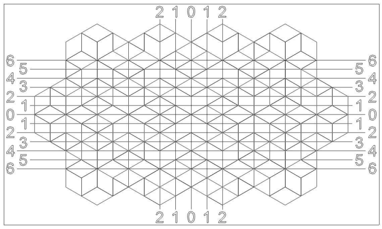


Figure 4

Figure 5

The tiling of the intersections can further be dissected by the perpendicular projected edges of the intersected solids. A similar phenomenon could be seen in the projection of the inner edges of the j-cube 3-models. In Figure 6 we have combined the grids of three and finally of all four horizontal intersections by $k = 6$. This is further dissected by the projected edges of the intersected solids.

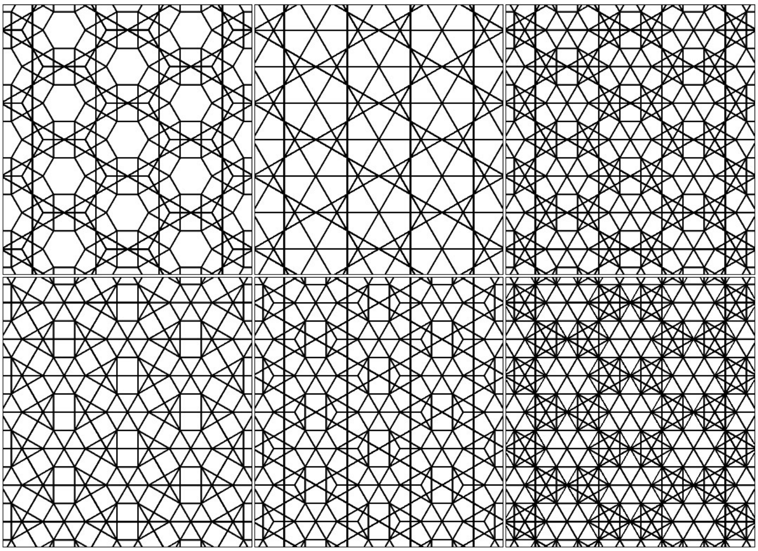


Figure 6

We can follow the construction of the space-filling mosaic based on the 3-model of the 8-cube by the next figures (Figure 7-8). The constructions way is similar to the case of the 6-cube. We can see in the last figure the places of the next repeating base-models.

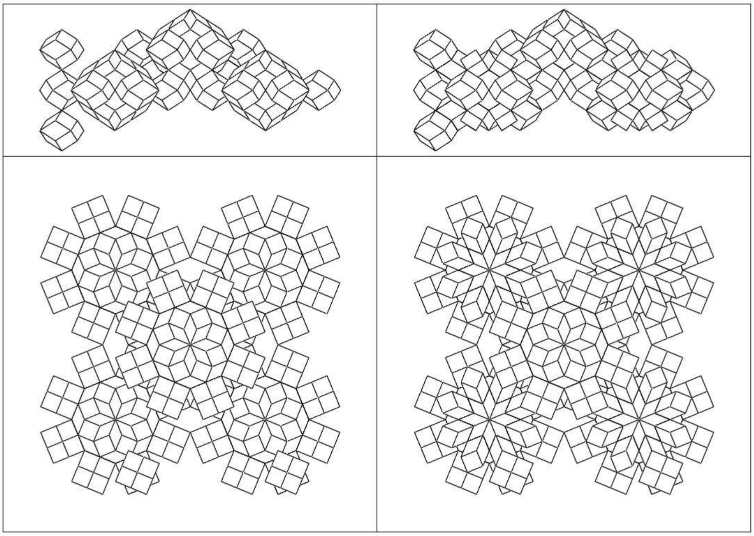


Figure 7

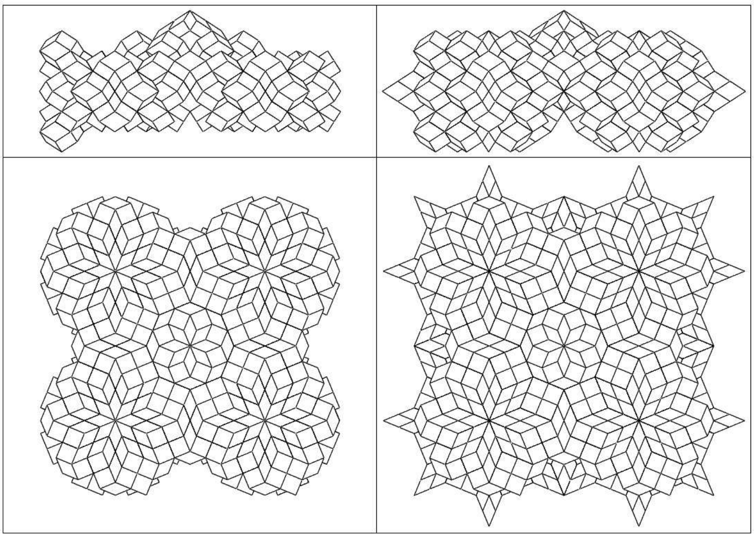


Figure 8

For odd k it is advisable to originate the construction from a $k-1$ sided polygon. In this case the k-th edge is perpendicular to the base plane but its upper endpoint must be in the common plane of the other edges' endpoints. We obtain a degenerated axonometric projection. Some sides fall into common planes but will not be identical. We gain on this way a 3-model which has similar symmetric-properties like this one in the case for even k. We can see the construction of the space-filling mosaic based on the 3-model of the 7-cube merged in one figure (Figure 9). The process is similair to the case of the 9-cube.

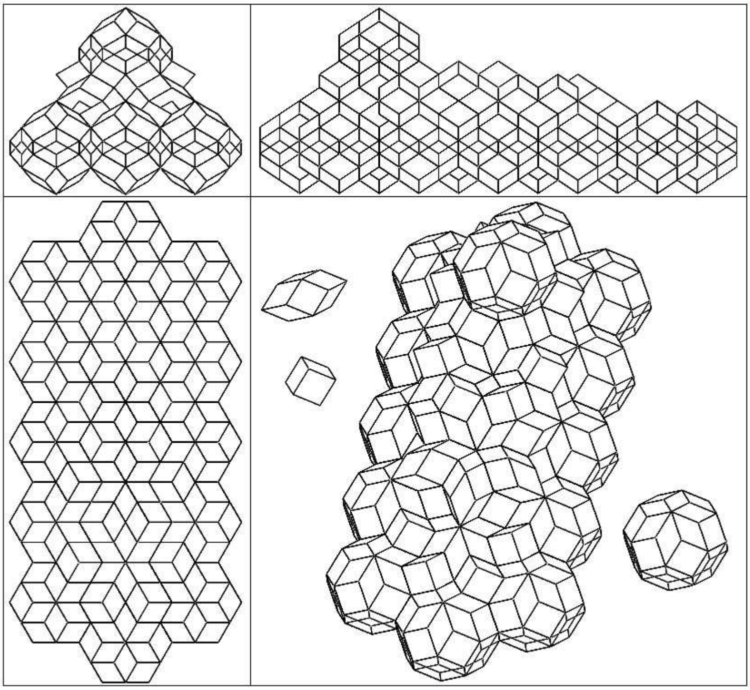


Figure 9

The construction is very simple based on the 3-model of the 5-cube. The repetition of this solid fills the space alone (Figure 10). As we know the 3-models of the 3-cubes originating from higher-dimensional spaces and the rhombic dodecahedrons (3-models of the 4-cubes) are also space-filling solids. The common part of the platonic hexahedron and octahedron fills the space too if its edges have an identical length. This polyhedron can be the shell of a 3-model of the 6-cube but the arrangement of the edges can be originated from the common part of the platonic hexahedron and octahedron if their edges halve each other (Figure 11).

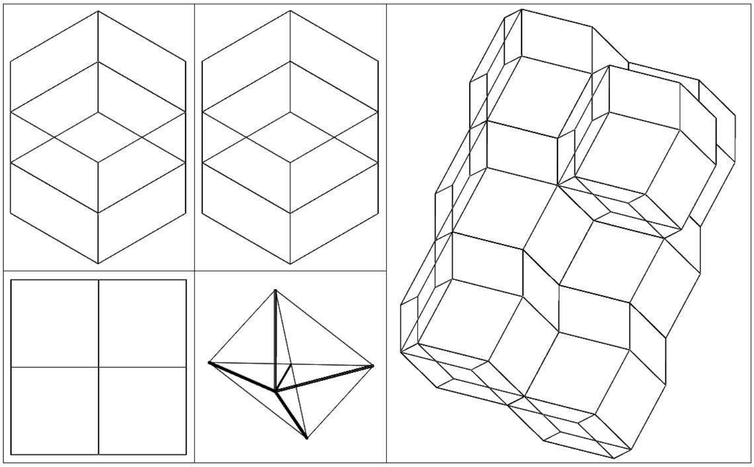


Figure 10

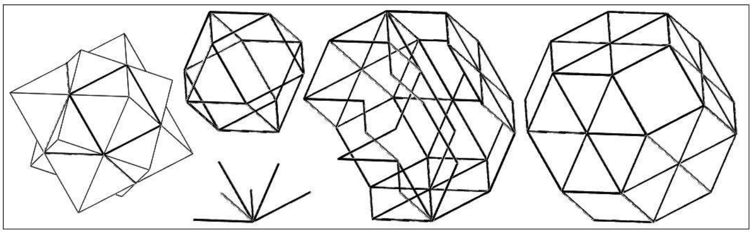


Figure 11

By these space-filling 3-models of the k-cubes is the question [4] even more justified: to what extent are these the isometric-axonometrical bildings of the higher-dimensional cubes created by a sequence of parallel projections. The Pohlke-theorem has surely limited validity in higher dimensions [2].

With the above methods two- and three-dimensional tilings based on the 3-models of k-cubes, can surely be made up to $k = 10$ and probably furthermore, too. These cases are just examined but not displayed yet in all details by the author.

The creation of the constructions and figures required for the paper was aided by the AutoCAD program and the Autolisp routines developed by the author.

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