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Visualization of the geodesic ball packings in the Nil geometry*

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Abstract

In the paper [7] the author investigated the geodesic balls of the **Nil** space and computed their volume, introduced the notion of a type of **Nil** lattices, **Nil** parallelepiped and the density of the lattice-like ball packing. Moreover, I have determined the densest packing with congruent geodesic balls where the ball centres form a type of integer lattices in the **Nil** space of Heisenberg matrices. To this the projective-affine interpretation of **Nil** geometry has been developed by E. Molnár in [1].

The careful analysis yields also the densest packing of density ≈ 0.78085 (larger than $\frac{\pi}{\sqrt{18}} \approx 0.74048\ldots$ for the analogous packing in the Euclidean space) with convex balls of special radius. It is remarkable that the kissing number of balls is 14 (in Euclidean space this number is 12). The projection of the optimal packing onto the Euclidean subplane of **Nil** will have hexagonal symmetry as the optimal circle arrangement of the Euclidean plane.

In this manner the geodesic lines, geodesic spheres and **Nil** lattices can be visualized on the Euclidean screen of computer, and this is the aim of the present paper. This visualization shows surprising phenomena as well. E.g. balls of radius $> \pi/2$ are not convex, balls of radius $> 2\pi$ deserve not to be defined.

1. On Nil geometry

The **Nil** geometry can be derived from the famous real matrix group $\mathbf{L}(\mathbf{R})$ discovered by Werner Heisenberg. The left (row-column) multiplication of Heisenberg matrices

$$\begin{pmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & a+x & c+xb+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{pmatrix} \tag{1.1}$$

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defines "translations" $\mathbf{L}(\mathbf{R}) = \{(x,y,z) : x,\ y,\ z \in \mathbf{R}\}$ on the points of the space $\mathbf{Nil} = \{(a,b,c) : a,\ b,\ c \in \mathbf{R}\}$. These translations are not commutative in general. The matrices $\mathbf{K}(z) \lhd \mathbf{L}$ of the form

$$\mathbf{K}(z) \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto (0,0,z) \tag{1.2}$$

constitute the one parametric centre, i.e. each of its elements commutes with all elements of $\mathbf{L}$. The elements of $\mathbf{K}$ are called *fibre translations*. $\mathbf{Nil}$ geometry can be projectively (affinely) interpreted by the "right translations" on points as the matrix formula

$$(1;a,b,c) \to (1;a,b,c) \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & x \\ 0 & 0 & 0 & 1 \end{pmatrix} = (1; x+a, y+b, z+bx+c) \tag{1.3}$$

shows, according to (1.1). Here we consider $\mathbf{L}$ as projective collineation group with right actions in homogeneous coordinates. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$,$E_1^\infty(\mathbf{e}_1)$,$E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Moreover, $\mathbf{y} = c\mathbf{x}$ with $0 < c \in \mathbb{R}$ (or $c \in \mathbb{R} \setminus \{0\}$) defines a point $(\mathbf{x}) = (\mathbf{y})$ of the projective 3-sphere $\mathcal{PS}^3$ (or that of the projective space $\mathcal{P}^3$ where opposite rays $(\mathbf{x})$ and $(-\mathbf{x})$ are identified). The dual system $\{(\boldsymbol{e}^i)\}$, $(\{\mathbf{e}^i\} \subset \boldsymbol{V}_4)$ describes the simplex planes, especially the plane at infinity $(\boldsymbol{e}^0) = E_1^\infty E_2^\infty E_3^\infty$, and generally, $\boldsymbol{v} = \boldsymbol{u}\frac{1}{c}$ defines a plane $(\boldsymbol{u}) = (\boldsymbol{v})$ of $\mathcal{PS}^3$ (or that of $\mathcal{P}^3$). Thus $0 = \mathbf{x}\boldsymbol{u} = \mathbf{y}\boldsymbol{v}$ defines the incidence of point $(\mathbf{x}) = (\mathbf{y})$ and plane $(\boldsymbol{u}) = (\boldsymbol{v})$, as $(\mathbf{x})\mathrm{I}(\boldsymbol{u})$ also denotes it. Thus $\mathbf{Nil}$ can be visualized in the affine 3-space $\mathbf{A}^3$ (so in $\mathbf{E}^3$) as well [3].

In [2] E. Molnár has shown that a rotation trough angle ω about the z-axis at the origin, as isometry of $\mathbf{Nil}$, keeping invariant the Riemann metric everywhere, will be a quadratic mapping in x, y to z-image $\overline{z}$ as follows:

$$\begin{gathered} \mathbf{r}(O,\omega) : (1;x,y,z) \to (1;\overline{x},\overline{y},\overline{z}); \\ \overline{x} = x\cos\omega - y\sin\omega, \quad \overline{y} = x\sin\omega + y\cos\omega, \\ \overline{z} = z - \frac{1}{2}xy + \frac{1}{4}(x^2-y^2)\sin 2\omega + \frac{1}{2}xy\cos 2\omega. \end{gathered} \tag{1.4}$$

The geodesic curves of the $\mathbf{Nil}$ geometry are generally defined as having locally minimal arc length between their any two (near enough) points. The equation systems of the parametrized geodesic curves $g(x(t), y(t), z(t))$ in our model can be determined by the general theory of Riemann geometry (see [3]): We can assume, that the starting point of a geodesic curve is the origin because of translations (1.1).

$$\begin{gathered} x(0) = y(0) = z(0) = 0; \quad \dot{x}(0) = c\cos\alpha, \ \dot{y}(0) = c\sin\alpha, \\ \dot{z}(0) = w; \ -\pi \leqslant \alpha \leqslant \pi. \end{gathered}$$

The arc length parameter s is introduced by

$$s = \sqrt{c^2 + w^2} \cdot t, \text{ where } w = \sin\theta,\ c = \cos\theta,\ -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2},$$

i.e. unit velocity can be assumed. The equation systems of a helix-like geodesic curve $g(x(t), y(t), z(t))$ if $0 < |w| < 1$ [3]:

$$\begin{gathered}
x(t) = \frac{2c}{w} \sin\frac{wt}{2} \cos\Big(\frac{wt}{2} + \alpha\Big), \quad y(t) = \frac{2c}{w} \sin\frac{wt}{2} \sin\Big(\frac{wt}{2} + \alpha\Big), \\
z(t) = wt \cdot \Big\{1 + \frac{c^2}{2w^2}\Big[\Big(1 - \frac{\sin(2wt + 2\alpha) - \sin 2\alpha}{2wt}\Big) + \\
+ \Big(1 - \frac{\sin(2wt)}{wt}\Big) - \Big(1 - \frac{\sin(wt + 2\alpha) - \sin 2\alpha}{2wt}\Big)\Big]\Big\} = \\
= wt \cdot \Big\{1 + \frac{c^2}{2w^2}\Big[\Big(1 - \frac{\sin(wt)}{wt}\Big) + \Big(\frac{1 - \cos(2wt)}{wt}\Big) \sin(wt + 2\alpha)\Big]\Big\}.
\end{gathered} \tag{1.5}$$

In the cases $w = 0$ the geodesic curve is the following:

$$x(t) = c \cdot t \cos\alpha, \quad y(t) = c \cdot t \sin\alpha, \quad z(t) = \frac{1}{2}\, c^2 \cdot t^2 \cos\alpha \sin\alpha. \tag{1.6}$$

Figure 1.b shows a geodesic curve in case $w = 0$ which is lying on the hyperbolic paraboloid surface $2z - xy = 0$. In Figure 2.a,b can be seen the path of a point P by a rotation trough angle 2π about the z-axis at the origin and two sites of geodesic curves OP at this rotation. The cases $|w| = 1$ are trivial: $(x, y) = (0, 0)$, $z = w \cdot t$.

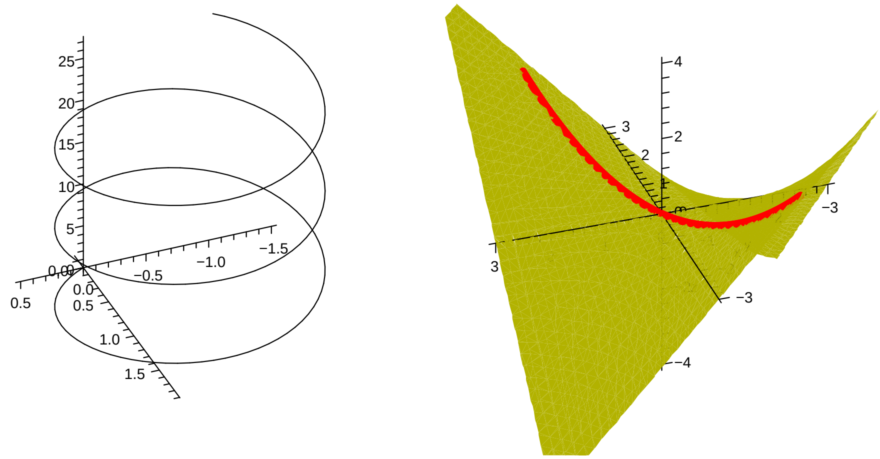

Figure 1: a, b

In Figure 1.a it can be seen a **Nil** geodesic curve with parameters $\alpha = \frac{\pi}{6}$, $\theta = \frac{\pi}{4}$, $t \in [0, 8\pi]$.

Definition 1.1. The distance $d(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the shortest geodesic curve from P_1 to P_2.

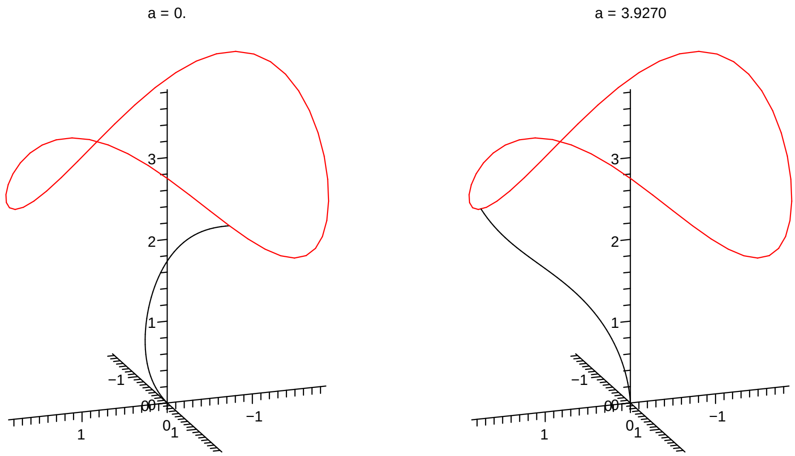

Figure 2: a, b

2. The geodesic ball

Definition 2.1. The geodesic sphere of radius R with centre at the point P_1 is defined as the set of all points P_2 in the space with the condition $d(P_1, P_2) = R$. Moreover, we require that the geodesic sphere is a simply connected surface without selfintersection in the **Nil** space.

The body of the geodesic sphere of centre P_1 and of radius R in the **Nil** space is called geodesic ball, denoted by $B_{P_1}(R)$, i.e. $Q \in B_{P_1}(R)$ iff $0 \leqslant d(P_1, Q) \leqslant R$.

Remark 2.2. Henceforth, typically we choose the origin as centre of the sphere and its ball, by the homogeneity of **Nil**.

Figure 3.a shows a geodesic sphere of radius $R = 4$ with centre at the origin. In Figure 3.b can be seen the cross-section of above ball lying in the coordinate plane $[x, z]$. In [7] we proved the following theorems:

Theorem 2.3. *The geodesic sphere and ball of radius R exists in the* **Nil** *space if and only if $R \in [0, 2\pi]$.*

In work [7] we have determined the volume of the geodesic ball of radius R by the following integral:

$$Vol(B(S)) = 2\pi \int_0^{\frac{\pi}{2}} X^2 \, \frac{\mathrm{d}\, Z}{\mathrm{d}\theta} \, \mathrm{d}\theta =$$

$$= 2\pi \int_0^{\frac{\pi}{2}} \Big(\frac{2\cos\theta}{\sin\theta} \sin\frac{(R\sin\theta)}{2} \Big)^2 \cdot \Big(-\frac{1}{2} \frac{R\cos^3\theta}{\sin^2\theta} + \frac{\cos\theta \sin(R\sin\theta)}{\sin\theta} + \\ + \frac{\cos^3\theta \sin(R\sin\theta)}{\sin^3\theta} - \frac{1}{2} \frac{R\cos^3\theta \cos(R\sin\theta)}{\sin^2\theta} \Big) \mathrm{d}\theta. \tag{2.1}$$

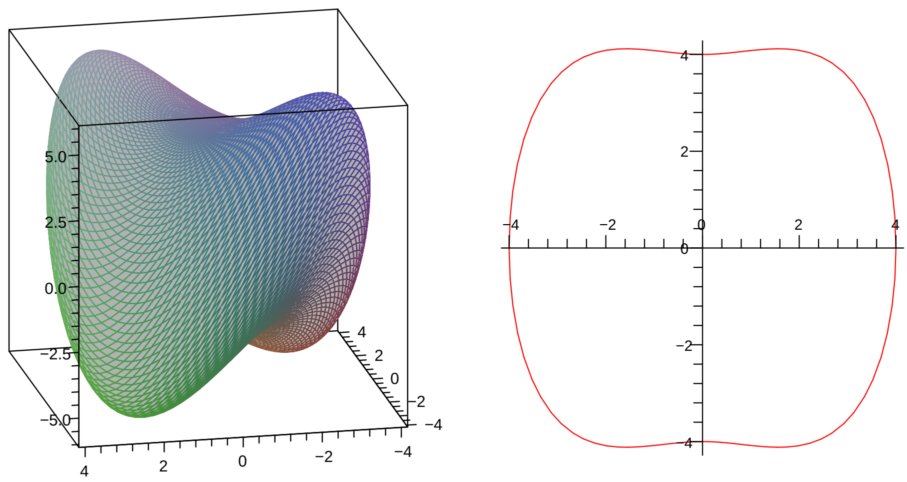

Figure 3: a, b

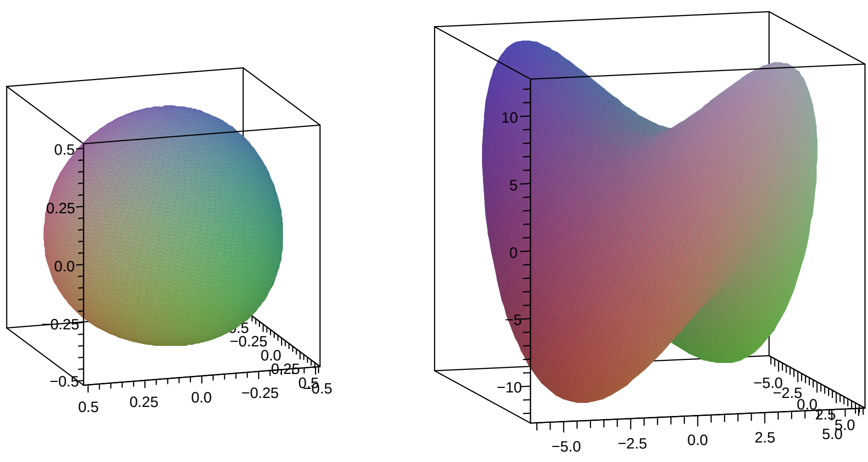

Figure 4: a, b

In [7] we have examined the convexity of the geodesic ball in Euclidean sense in our affine model and we have obtained the following theorem (see Figure 4.a: $R = 0.5$, b: $R = 6$):

Theorem 2.4. *The geodesic* **Nil** *ball $B(S(R))$ is convex in affine-Euclidean sense in our model if and only if $R \in [0, \frac{\pi}{2}]$.*

3. The discrete translation group L(Z, k)

We consider the **Nil** translations defined in (1.1) and (1.3) and choose two arbitrary translations

$$\tau_1 = \begin{pmatrix} 1 & t_1^1 & t_1^2 & t_1^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_1^1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } \tau_2 = \begin{pmatrix} 1 & t_2^1 & t_2^2 & t_2^3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t_2^1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.1}$$

now with upper indices for coordinate variables. We define the translation $(\tau_3)^k$, $(k \in \mathbb{N}, \geqslant 1)$ by the following commutator:

$$(\tau_3)^k = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1 = \begin{pmatrix} 1 & 0 & 0 & -t_2^1 t_1^2 + t_1^1 t_2^2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

If we take integers as coefficients, their set is denoted by $\mathbb{Z}$, then we generate the discrete group $(\langle \tau_1, \tau_2 \rangle, k)$ denoted by $\mathbf{L}(\tau_1, \tau_2, k)$ or by $\mathbf{L}(\mathbb{Z}, k)$. (See also Remark 2.2)

We know (see e.g. [3]) that the orbit space $\mathbf{Nil}/\mathbf{L}(\mathbb{Z}, k)$ is a compact manifold, i.e. a **Nil** space form.

Definition 3.1. The **Nil** point lattice $\Gamma_P(\tau_1, \tau_2, k)$ is a discrete orbit of point P in the **Nil** space under the group $\mathbf{L}(\tau_1, \tau_2, k) = \mathbf{L}(\mathbb{Z}, k)$ with an arbitrary starting point P for all $(k \in \mathbb{N}, k \geqslant 1)$.

Remark 3.2. 1. For simplicity we have chosen the origin as starting point, by the homogeneity of **Nil**.

2. We can assume that $t_1^2 = 0$, i.e. the image of the origin by the translation τ_1 lies on the plane $[x, z]$.

In the following we investigate the most important case $k = 1$ where τ_3 corresponds to the fibre translation $\tau_3 = \tau_2^{-1}\tau_1^{-1}\tau_2\tau_1$.

We illustrate the action of $\mathbf{L}(\mathbb{Z}, 1)$ on the **Nil** space in Figure 5. We consider a non-convex polyhedron $\mathcal{F} = OT_1T_2T_3T_{12}T_{21}T_{23}T_{213}T_{13}$, in Euclidean sense, which is determined by translations τ_1, τ_2, τ_3. This polyhedron determine a solid $\widetilde{\mathcal{F}}$ in the **Nil** space whoose images under $\mathbf{L}(\mathbb{Z}, 1)$ fill the **Nil** space just once, i.e. without gap and overlap.

Analogously to the Euclidean integer lattice and parallelepiped, the solid $\widetilde{\mathcal{F}}$ *can be called* **Nil** *parallelepiped.*

$\widetilde{\mathcal{F}}$ is a *fundamental domain* of $\mathbf{L}(\mathbb{Z}, 1)$. The homogeneous coordinates of the vertices of $\widetilde{\mathcal{F}}$ can be determined in our affine model by the translations (3.1) and (3.2) with the parameters $t_i^j,\ i \in \{1,2\},\ j \in \{1,2,3\}$ (see Figure 5 and (3.3)).

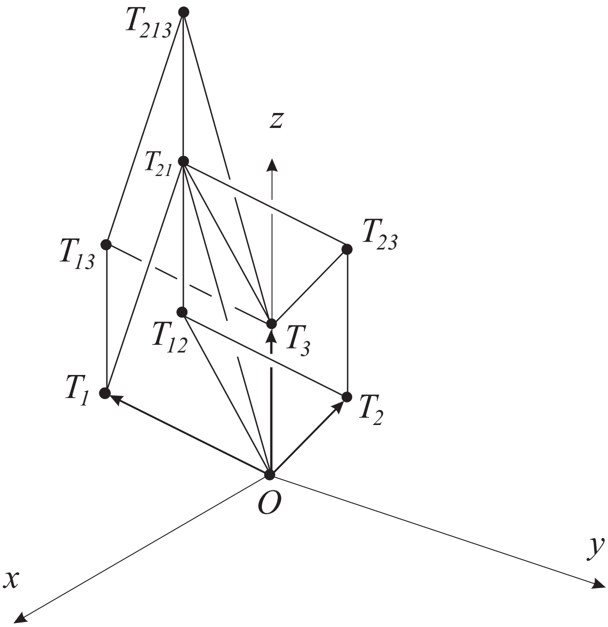

Figure 5

$$\begin{gathered} T_1(1, t_1^1, 0, t_1^3),\ T_2(1, t_2^1, t_2^2, t_2^3),\ T_3(1, 0, 0, t_1^1 t_2^2), \\ T_{13}(1, t_1^1, 0, t_1^1 t_2^2 + t_1^3),\ T_{12}(1, t_1^1 + t_2^1, t_2^2, t_2^3 + t_1^3), \\ T_{21}(1, t_1^1 + t_2^1, t_2^2, t_1^1 t_2^2 + t_1^3 + t_2^3),\ T_{23}(1, t_2^1, t_2^2, t_2^3 + t_1^1 t_2^2), \\ T_{213} = T_{231}(1, t_1^1 + t_2^1, t_2^2, 2t_1^1 t_2^2 + t_1^3 + t_2^3). \end{gathered} \tag{3.3}$$

We have determined in [7] the volume of $\widetilde{\mathcal{F}}$ by $\det(\overrightarrow{OT}_1, \overrightarrow{OT}_2, \overrightarrow{OT}_3) = (t_1^1 \cdot t_2^2)^2$ from (3.3) or by the following integral:

$$Vol(\widetilde{\mathcal{F}}) = \int_0^{t_2^2} \int_0^{t_1^1} |t_1^1 \cdot t_2^1| \, \mathrm{d}x \mathrm{d}y = (t_1^1 \cdot t_2^2)^2. \tag{3.4}$$

From this formula it can be seen that the volume of the **Nil** parallelepiped depends on two parameters, i.e. on its projection onto the $[x, y]$ plane.

4. Lattice-like geodesic ball packings

Let $\mathcal{B}_\Gamma(R)$ denote a geodesic ball packing of **Nil** space with balls $B(R)$ of radius R where their centres give rise to a **Nil** point lattice $\Gamma(\tau_1, \tau_2, 1)$. $\widetilde{\mathcal{F}}_0$ is an arbitrary **Nil** parallelepiped of this lattice (see (3.1),(3.2)). The images of $\widetilde{\mathcal{F}}_0$ by our discrete translation group $\mathbf{L}(\tau_1, \tau_2, 1)$ covers the **Nil** space without overlap. For the density of the packing it is sufficient to relate the volume of the optimal ball to that of the solid $\widetilde{\mathcal{F}}_0$. Analogously to the Euclidean case it can be defined the density $\delta(R, \tau_1, \tau_2, 1)$ of the lattice-like geodesic ball packing $\mathcal{B}_\Gamma(R)$:

Definition 4.1.

$$\delta(R, \tau_1, \tau_2, 1) := \frac{Vol(\mathcal{B}_\Gamma(R) \cap \widetilde{\mathcal{F}}_0)}{Vol(\widetilde{\mathcal{F}}_0)}, \tag{4.1}$$

if the balls do not overlap each other.

Remark 4.2. By definition of the **Nil** lattice $\mathbf{L}(\tau_1, \tau_2, 1)$(see Definition 3.1) the orbit space $\mathbf{Nil}/\mathbf{L}(\tau_1, \tau_2, 1)$ is a compact **Nil** manifold, and (see Section 2),

$$Vol(\mathcal{B}_\Gamma(R) \cap \widetilde{\mathcal{F}}_0) = Vol(B(S(R))).$$

The optimal lattice-like ball packing

We look for such an arrangement $\mathcal{B}_\Gamma(R)$ of balls $B(R)$, (see Figure 5) where the following equations hold:

$$\begin{aligned}
&(a)\ \ d(O, T_1) = 2R = d(T_1, T_3),\\
&(b)\ \ d(O, T_2) = 2R = d(T_2, T_3),\\
&(c)\ \ d(T_1, T_2) = 2R,\\
&(d)\ \ d(O, T_3) = 2R.
\end{aligned} \tag{4.2}$$

Here d is the distance function in the **Nil** space (see Definition 1.1). The equations (a) and (b) mean that the ball centres T_1 and T_2 lie on the equidistant geodesic surface of the points O and T_3 which is a hyperbolic paraboloid (see (1.6) and Figure 6) in our model with equation

$$2z - xy = 2R.$$

By continuity of the distance function it follows, that there is a (unique) solution of the equation system (4.2). We have denoted by $\mathcal{B}^{opt}_\Gamma(R_{opt})$ the geodesic ball packing of the balls $B(R_{opt})$ which satisfies the above equation system. We get the following solution by systematic approximation, where the computations were carried out by *Maple V Release 10* up to 30 decimals:

$$\begin{gathered}
t_1^{1,opt} \approx 1.30633820,\ t_1^{3,opt} = R_{opt},\ R_{opt} \approx 0,73894461;\\
t_2^{1,opt} \approx 0,65316910,\ t_2^{2,opt} \approx 1,13132206,\ t_2^{3,opt} \approx 1.10841692,\\
T_1^{opt} = (1, t_1^{1,opt}, 0, t_1^{3,opt}),\ \ T_2^{opt} = (1, t_2^{1,opt}, t_2^{2,opt}, t_2^{3,opt}).
\end{gathered} \tag{4.3}$$

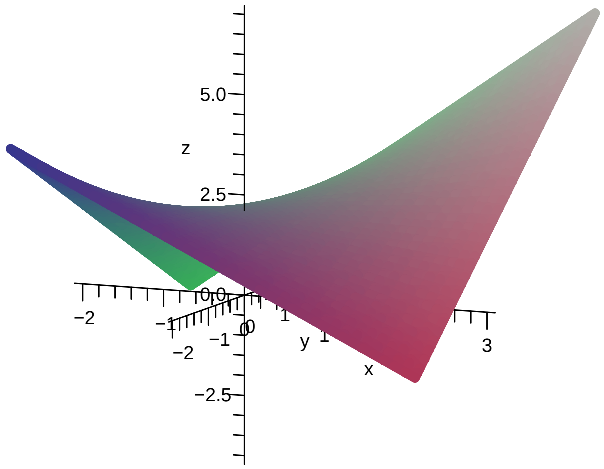

Figure 6

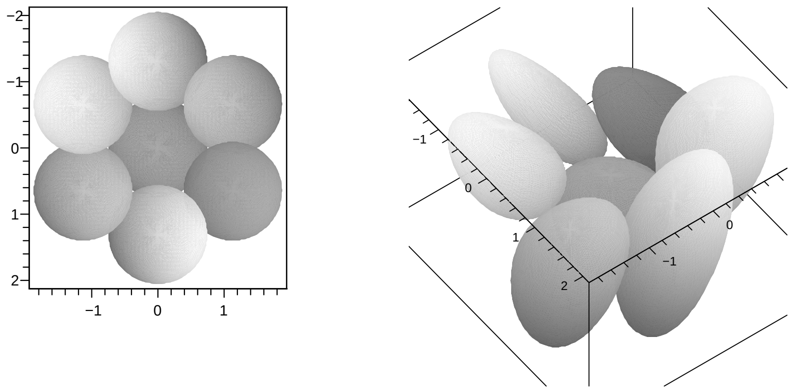

Figure 7: a, b

The geodesic ball packing $\mathcal{B}_\Gamma^{opt}(R_{opt})$ can be realized in **Nil** space because by Theorem 2.4 a ball of radius $R_{opt} \approx 0,73894461$ is convex in Euclidean sense and this packing can be generated by the translations $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt}, 1)$ where τ_1^{opt} and τ_2^{opt} are given by the coordinates $t_i^{j,opt}$ $i = 1,2;$ $j = 1,2,3$ (see (4.3) and (3.1)). Thus we obtain the neigbouring balls around an arbitrary ball of the packing $\mathcal{B}_\Gamma^{opt}(R_{opt})$ by the lattice $\Gamma(\tau_1^{opt}, \tau_2^{opt}, 1)$, the kissing number of the balls is 14. Figure 8.a,b show the typical arrangement of some balls from $\mathcal{B}_\Gamma^{opt}(R_{opt})$ in our model. We have ball "columns" in z-direction and in regular hexagonal projection onto the $[x, y]$-plane (see Figure 7.a,b). The fundamental domain $\widetilde{\mathcal{F}}_0^{opt}$

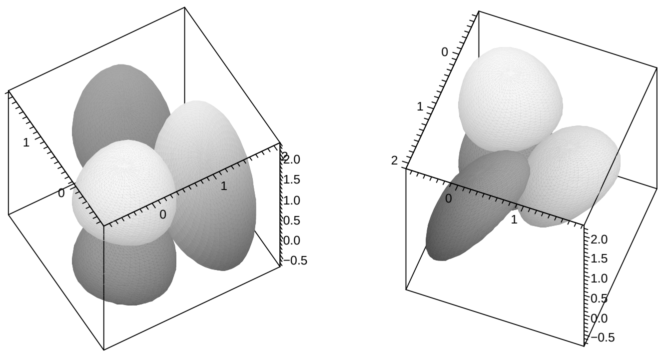

Figure 8: a, b

of the discrete translation group $\mathbf{L}_{opt}(\tau_1^{opt}, \tau_2^{opt})$ is a **Nil** parallelepiped of the above determined **Nil** lattice $\Gamma(\tau_1^{opt}, \tau_2^{opt}, 1)$. By formulas (2.1), (3.4) and by definition 4.1 we can compute the density of this ball packing:

$$\begin{gathered} Vol(\widetilde{\mathcal{F}}_0^{opt}) \approx 2,18415656, \quad Vol(\mathcal{B}_\Gamma(R_{opt}) \cap \widetilde{\mathcal{F}}_0^{opt}) \approx 1,70548775, \\ \delta(R_{opt}, \tau_1^{opt}, \tau_2^{opt}, 1) := \frac{Vol(\mathcal{B}_\Gamma(R_{opt}) \cap \widetilde{\mathcal{F}}_0^{opt})}{Vol(\widetilde{\mathcal{F}}_0^{opt})} \approx 0,78084501. \end{gathered} \tag{4.4}$$

In [7] we have proved the following Theorem:

Theorem 4.3. *The ball arrangement $\mathcal{B}_\Gamma^{opt}(R_{opt})$ given in formulas (4.3), (4.4) provides the optimal lattice-like geodesic ball packing to the lattices $\Gamma(\tau_1, \tau_2, 1)$ in the* **Nil** *space.*

Analogous questions in **Nil** geometry or, in general, in other homogeneous Thurston [6] geometries are on our program with E. Molnár and I. Prok.

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