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Simple digital objects on $\mathbb{Z}^2$*

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Abstract

In this paper we present some simple digital geometrical concepts. The used spaces are the square grid ($\mathbb{Z}^2$) with the well-known metric distances based by neighbourhood relations 4 and 8-neighbours. We define and analyse some properties of the digital line segments, lines, circles, parabolas, hyperbolas and ellipses.

Keywords: Digital geometry, digital distances

MSC: 52Cxx: Discrete geometry, 68U05 Computer graphics; computational geometry

1. Introduction

The digital geometry is an important part of digital image processing. In this paper we will use the square grid with distances based on 4-neighbourhood and on 8-neighbourhood relations. We will use the notations $(\mathbb{Z}^2, 4)$ and $(\mathbb{Z}^2, 8)$, respectively. We will use the distance functions based on the possible shortest paths between points using only the neighbourhood criteria of the plane in each step.

The purpose of this paper is to describe some simple geometrical objects of these digital planes. Our aim is not to have digital objects which can represent Euclidean object (it is the matter of the computer graphic). Our definitions are based on digital distances and some characteristic Euclidean properties of the objects. We will refer the pairs $p = (p_1, p_2) \in \mathbb{Z}^2$ as points.

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2. Segments and lines

In this section we will define two natural concepts. The concept of the point was an obvious choice in the digital plane. Now we are using the following Euclidean property of the segments and lines: "they contain points of the shortest path between two points".

Since in the digital plane it can be several shortest paths between two given points $p, q \in \mathbb{Z}^2$ we will chose some special ones.

First let us see the case of 4-neighbourhood.

Definition 2.1. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ be two different points in $(\mathbb{Z}^2, 4)$. A path $\Pi_4(p, q)$ with minimal length is called *segment between p and q* if for all $r = (r_1, r_2) \in \Pi_4(p, q)$ at least one of the following conditions hold: $r_1 = p_1$, $r_1 = q_1$, $r_2 = p_2$, $r_2 = q_2$.

This definition gives us the two extreme paths between the points, namely the first is given by the points $(p_1 + \text{sgn}(q_1 - p_1)t, p_2)$ where $t = 0, \dots, |q_1 - p_1|$; $(q_1, p_2 + \text{sgn}(q_2 - p_2)t)$ with $t = 0, \dots, |q_2 - p_2|$ and the second has the points $(p_1, p_2 + \text{sgn}(q_2 - p_2)t)$ with $t = 0, \dots, |q_2 - p_2|$ and $(p_1 + \text{sgn}(q_1 - p_1)t, q_2)$ with $t = 0, \dots, |q_1 - p_1|$. When $p_1 = q_1$ or $p_2 = q_2$ then the above segments are identical.

Figure 1: Segments and lines connecting 2 points in $(\mathbb{Z}^2, 4)$

These segments can be extended to lines of different types as follows.

Definition 2.2. Let $p = (p_1, p_2)$ a point in $\mathbb{Z}^2$ and $s_1, s_2 \in \{-1, 1\}$. We say that $L(p, s_1, s_2)$ is a *(digital) line* of the space $(\mathbb{Z}^2, 4)$ if it is constructed by the following points $(p_1 + s_1 t, p_2)$ and $(p_1, p_2 + s_2 t)$ where $t \in \mathbb{N}$. Moreover, $L^h(p_2)$ is called a *horizontal line* if it has the points (t, p_2) for t in $\mathbb{Z}$ and $L^v(p_1)$ is a *vertical line* analogously.

The next results show the differences between the Euclidean plane and $(\mathbb{Z}^2, 4)$.

Lemma 2.3. *If p, q in $\mathbb{Z}^2$ and $p_1 \neq q_1$, $p_2 \neq q_2$ then there exist exactly two lines which contains both the points p and q. They are*

$$L((p_1, q_2), \text{sgn}(q_1 - p_1), \text{sgn}(p_2 - q_2)) \quad \textit{and} \quad L((q_1, p_2), \text{sgn}(p_1 - q_1), \text{sgn}(q_2 - p_2)).$$

Lemma 2.4. *If p and q are different points in $\mathbb{Z}^2$ and $p_1 = q_1$ or $p_2 = q_2$ then there are one horizontal or vertical line and infinitely many other lines which contain q and p. In the first case they are: $L^v(p_1)$ and $L((p_1, t), s_1, \text{sgn}(p_2 - t))$ where*

$s_1 \in \{-1, 1\}$ *and* t *is an integer with the property* $\text{sgn}(p_2 - t)\text{sgn}(q_2 - t) \geqslant 0$. *In the second case the horizontal line is* $L^h(p_2)$ *and the other lines are* $L((t, p_2), \text{sgn}(p_1 - t), s_2)$ *where* $s_2 \in \{-1, 1\}$ *and* $\text{sgn}(p_1 - t)\text{sgn}(q_1 - t) \geqslant 0$.

The concepts defined above has the similar property as the Euclidean segments and lines that we detail below. For any two points of a line there is a segment such that it connects the points and it is a part of the line. Note here that using another definitions of digital segments and lines this property may not hold for any two points of the digital lines.

Definition 2.5. We say that *two lines are parallel* if all the points of either line have the same distance from the other line.

Note, that it is not true for both lines that all of the points have the same distance from the other line.

Now, let us analyse the fifth postulate of Euclid. It works in the following way in $(\mathbb{Z}^2, 4)$.

Theorem 2.6. *If* L_1 *is a line and* p *is a point in* $\mathbb{Z}^2$ *outside of* L_1*, then there exists a uniquely determined line which is parallel with* L_1 *and contains* p*.*

Now, let us define the analogous concepts in $(\mathbb{Z}^2, 8)$. Let us choose the segments and lines on this plane based on the extremal shortest paths as well.

In general we can have two kinds of segments between points according to the parities of the coordinate values of the points.

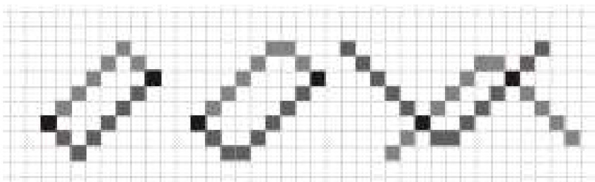

Figure 2: Segments and lines connecting 2 points in $(\mathbb{Z}^2, 8)$

Definition 2.7. Let $p = (p_1, p_2)$ and $q = (q_1, q_2)$ are arbitrary points in $(\mathbb{Z}^2, 8)$. We say that the minimal path $\Pi_8(p, q)$ is a *segment connecting* p *and* q if

$$|p_1 - r_1| = |p_2 - r_2| \quad \text{or} \quad |q_1 - r_1| = |q_2 - r_2|$$

hold for all $r = (r_1, r_2) \in \Pi_8(p, q)$.

These segments can be extended to lines in a similar way than in case of $(\mathbb{Z}^2, 4)$.

3. Second order objects in $(\mathbb{Z}^2, 4)$

The digital circles on $\mathbb{Z}^2$ are well examined. In this section we also describe the digital ellipses, hyperbolas and parabolas using the previous concepts of points and lines.

First the plane $(\mathbb{Z}^2, 4)$ is used. Let p be a given point in $\mathbb{Z}^2$. It is known that the circle with center p and radius r has the points $(p_1 + s_1 t, p_2 + s_2(r - t))$ where $0 \leqslant t \leqslant r$ and $s_1, s_2 \in \{-1, 1\}$. These circles are built up from segments of $(\mathbb{Z}^2, 8)$ and have diamond shapes. We can also determine the points of an ellipse.

Definition 3.1. Let p and q be different points in $(\mathbb{Z}^2, 4)$ and r is a natural number. The *digital ellipse* $E(p, q; r)$ is the set of points z which has the property $d(z, p) + d(z, q) = r$.

We remark that if $r < d = d(p, q)$ or d and r has not the same parity then $E(p, q; r)$ has no points, hence it is sufficient to deal with ellipses of the form $E(p, q; d + 2h)$. Furthermore, the translation invariance and the symmetry of the metric allows us to suppose that one of the focuses of the ellipse is the origin and the other has nonnegative coordinates.

Theorem 3.2. *If $q \in (\mathbb{Z}^2, 4)$ is a point with $q_1, q_2 \geqslant 0$, $d = d(0, q)$ and h is a positive integer, then the points of $E(0, q; d + 2h)$ are exactly the following ones:*

1.	$(-t, t - h)$,	$t = 0, 1, \dots h$,
2.	$(-t, q_2 + h - t)$,	$t = 0, 1, \dots h$,
3.	$(-h, t)$,	$t = 0, 1, \dots q_2$,
4.	$(t, -h)$,	$t = 0, 1, \dots q_1$,
5.	$q - v$,	*for all previous points v in cases* 1., 2., 3., 4.

Proof. The point (x, y) is an element of the ellipse $E(0, q; d + 2h)$ if and only if

$$|x| + |y| + |x - q_1| + |y - q_2| = q_1 + q_2 + 2h.$$

It is obvious, that if $v = (x, y)$ is a point of the ellipse then $q - v$ also is an element of E. If $x \leqslant 0$ and $y \leqslant 0$ then $x - q_1$ and $y - q_2$ are also nonnegative, so we have for the above equation that $-x - y = h$, which gives us the case 1. in the theorem. If $x \leqslant 0$ and $y \geqslant q_2$ then $y - x = q_2 + h$, which is case 2. If $x \leqslant 0$, and $0 \leqslant y \leqslant q_2$ then $-x = h$, and similarly, if $y \leqslant 0$ and $0 \leqslant x \leqslant q_1$ then $y = -h$, these are case 3. and 4. The remaining cases give points of the form $q - v$, where v satisfies one of the previous conditions. On the other hand, it is easy to see that for the points in the theorem the equation of the ellipse holds. $\square$

Now we characterize the digital hyperbolas on the plane $(\mathbb{Z}^2, 4)$.

Definition 3.3. Let p and q are different points in $(\mathbb{Z}^2, 4)$ and r is a natural number. The set of all points v which have the property $|d(v, p) - d(v, q)| = r$ is called the *digital hyperbola* $H(p, q; r)$.

Theorem 3.4. *Let* $q \in (\mathbb{Z}^2, 4)$, $q_1, q_2 \geqslant 0$, $d = d(0,q)$ *and* $h > d/2$ *a positive integer. The points of the digital hyperbola* $H(0, q; d-2h)$ *are exactly the following ones:*

1. $(t, h-t)$, $\quad \max\{0, h-q_2\} \leqslant t \leqslant \min\{q_1, h\}$
2. (a) *if* $h > q_1$, *then* $(t, h-q_1)$, $t \geqslant q_1$,
   (b) *if* $h < q_1$, *then* (h, t), $t \leqslant 0$,
   (c) *if* $h = q_1$, *then* (t_1, t_2), $t_1 \geqslant q_1$, $t_2 \leqslant 0$,
3. (a) *if* $h > q_2$, *then* $(h-q_2, t)$, $t \geqslant q_2$,
   (b) *if* $h < q_2$, *then* (t, h), $t \leqslant 0$,
   (c) *if* $h = q_2$, *then* (t_1, t_2), $t_2 \geqslant q_2$, $t_1 \leqslant 0$,
4. $q - v$, *for all previous points* v *in cases* 1., 2., 3.

Proof. The equation of the hyperbola's branch which is closer to the origin:

$$|x-q_1| + |y-q_2| - |x| - |y| = q_1 + q_2 - 2h.$$

It is easy to see that the point $v = (x, y)$ satisfies this equation then $q - v$ is an element of the other branch of the hyperbola. If $x \geqslant 0$ and $y \geqslant 0$, then the above equation can be written in the form

$$2h = [(x+q_1) - |x-q_1|] + [(y+q_2) - |y-q_2|],$$

that is $h = \min\{x, q_1\} + \min\{y, q_2\}$ and here $h = q_1 + q_2 = d$ is not possible. Hence if $x > q_1$ then $h = y + q_1$, and if $h \geqslant q_1$ then we have the case 2.(a), and similarly, $y \geqslant q_2$ gives the case 3.(a). When $x < q_1$ and $y < q_2$, then $h = x + y$, it is the case 1. If $x \geqslant 0$, $y \leqslant 0$ then $|x - q_1| - x = q_1 - 2h$, that means $h = min\{q_1, x\}$, and we get the 2.(b) and 2.(c). Transposing the role of the coordinates x and y we have 3.(b) and 3.(c). Finally, if $x \leqslant 0$ and $y \leqslant 0$ then $h = 0$, and this is a contradiction. $\square$

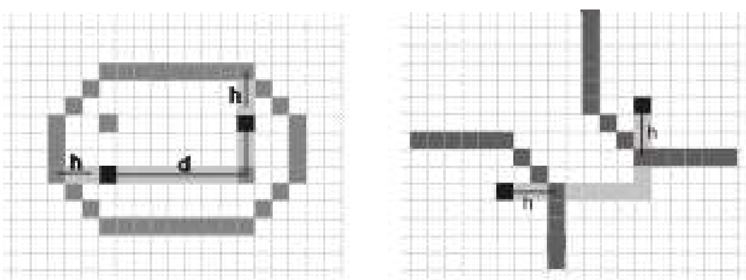

Figure 3: An ellipse and a hyperbola

Note that the lines connecting the focus points play the similar role as the axes play at the objects in the Euclidean plane as it can be seen on the Figure 3.

The third important class of second order objects are the parabolas.

Definition 3.5. Let L be a line and p be a point in $(\mathbb{Z}^2, 4)$. The *digital parabola* $P(L, p)$ is the set of points z which has the property $d(z, p) = d(z, L)$.

As we have seen, there are different types of lines in $(\mathbb{Z}^2, 4)$, therefore various kinds of parabolas are possible. Here, for simplicity, we only give the points of parabolas with directrix and axis as the coordinate axises.

Theorem 3.6. *Let $p = (0, p_2) \in (\mathbb{Z}^2, 4)$ and $p_2 > 0$. The points of the parabola $P(p, L^h(0)$ are those which have integer coordinates among the following ones:*

1. $(\pm p_2, t)$, $\quad t \geqslant p_2$,
2. $(\pm(p_2 - 2t), p_2 - t)$, $\quad t = 0, \ldots, \left\lfloor \dfrac{p_2}{2} \right\rfloor$.

Proof. Note that the distance of the point (x, y) and the line $L^h(p_2)$ is $|y - p_2|$. It is clear, that if a point (x, y) satisfies the equation of the parabola, which is $|x| + |y - p_2| = |y|$, then $(-x, y)$ is also an element of P. The case $y \leqslant 0$ leads to the contradiction $|x| = -p_2$, so it is sufficient to investigate the case $x \geqslant 0$ and $y \geqslant 0$. Now the equations has the form $x + |y - p_2| = y$, so if $y \geqslant p_2$ then $x = p_2$, and when $y \leqslant p_2$ then $y = \dfrac{x + p_2}{2}$. □

Similar characterizations are possible in cases of different lines, some examples can be seen in Figure 4.

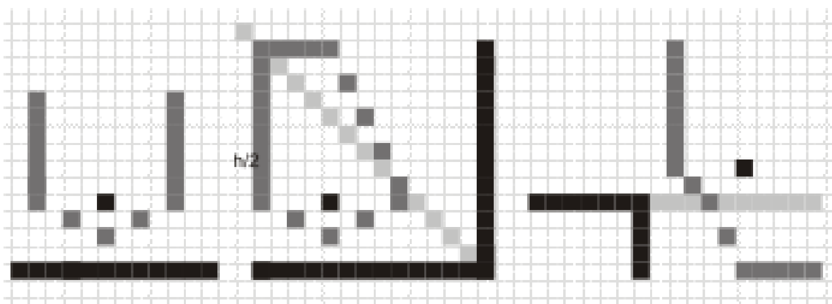

Figure 4: Possible parabolas in $(\mathbb{Z}^2, 4)$

4. Second order objects in $(\mathbb{Z}^2, 8)$

The digital circles on $(\mathbb{Z}^2, 8)$ have square shapes and they built up by segments of $(\mathbb{Z}^2, 4)$. With center p and radius r it contains points $(p_1 \pm sr + (1 - s)t, p_2 + st \pm (1 - s)r)$ where $-r \leqslant t \leqslant r$ and $s \in \{0, 1\}$.

We have similar characterization results for ellipses, hyperbolas and parabolas on the plane $(\mathbb{Z}^2, 8)$ as well. Using the notation of definitions 3.1 and 3.3, it is not necessary for r and d to have the same parity, but certain parts of these objects may be missing.

Theorem 4.1. *Let* $q \in (\mathbb{Z}^2, 8)$, $q_1 \geqslant q_2 \geqslant 0$, $d = d(0, q)$ *and* h *be a positive integer. The points of the ellipse* $E(0, q; d + h)$ *are the following ones (requiring that the coordinates are integers):*

1. $(-h/2, t)$, $\quad -h/2 \leqslant t \leqslant h/2$,
2. $(t, (q_1 + q_2 + h)/2)$, $\quad (q_1 + q_2 - h)/2 \leqslant t \leqslant (q_1 + q_2 + h)/2$,
3. $(t - h, t)$, $\quad h/2 \leqslant t \leqslant (q_1 + q_2 + h)/2$,
4. $(t, -h - t)$, $\quad -h/2 \leqslant t \leqslant (q_1 - q_2 - h)/2$,
5. $q - v$, $\quad$ *for all previous points* v *in cases* $1., 2., 3., 4.$

Theorem 4.2. *Let* $q \in (\mathbb{Z}^2, 8)$, $q_1, q_2 \geqslant 0$, $d = d(0, q)$ *and* $0 < h < d$ *be a natural number. The points of the hyperbola* $H(0, q; d - h)$ *are the following ones:*

1. $(h/2, t)$, $\quad \max\{-h/2, q_2 - q_1 + h/2\} \leqslant t \leqslant h/2$,
2. $(h - t, t)$, $\quad t \geqslant h/2$,
3. (a) *if* $h > q_1 - q_2$, *then* $(t, h + q_2 - q_1 - t)$, $h/2 \leqslant t$,
   (b) *if* $h < q_1 - q_2$, *then* $(t, t - h)$, $t \leqslant h/2$,
   (c) *if* $h = q_1 - q_2$, *then* (t_1, t_2), $t_2 \leqslant -h/2$, $t_2 + h \leqslant t_1 \leqslant -t_2$,
4. $q - v$, $\quad$ *for all previous points* v *in cases* $1., 2., 3.$

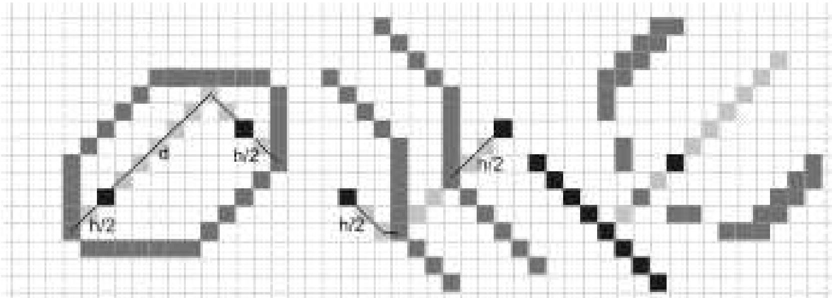

Figure 5: Possible ellipse, hyperbola and parabola in $(\mathbb{Z}^2, 8)$

Theorem 4.3. *Let* $(p, p) \in (\mathbb{Z}^2, 8)$ *and* $L(0) = \{(s, -s) | s \in \mathbb{Z}\}$. *The points of the parabola* $P(p, L(0))$ *are the following ones (requiring that the coordinates are integers):*

1. $(t, (2p - t)/3)$, $\quad \lceil p/2 \rceil \leqslant t \leqslant 2p - 1$,
2. $(t, (2p - t - 1)/3)$, $\quad \lceil p/2 \rceil \leqslant t \leqslant 2p - 1$,
3. $(t, t - 2p)$, $\quad 2p \leqslant t$,
4. $(t, t - 2p - 1)$, $\quad 2p + 1 \leqslant t$,
5. (q_2, q_1), $\quad$ *for all previous points* (q_1, q_2) *in cases* $1., 2., 3., 4.$

Finally we show some examples of these digital objects in Figure 5.

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