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Some aspects of clothoids

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Abstract

Clothoids are curves used especially for road design. Their main property is, that the curvature varies proportionally with the arc length. They cannot be described exactly in an analytic sense because the clothoid is based on Fresnel integrals. In order to implement the clothoid in CAD systems certain approximation methods are used. We choose a polynomial approximation for the 2D-clothoid based on s-power series as this method is very accurate and simultaneously has a low degree of the Bézier polynomials. We describe offsets of 2D-clothoids, defined by its normal and study geometrical attributes of these offset curves.

Furthermore methods for a three dimensional approximation of the clothoid are proposed and discussed, focussing on the behaviour of curvature and arc length of the spatial curve.

Keywords: 3D clothoid curve, Bézier curve, Hermite approximation

1. Definition of clothoids

Clothoids are defined using Fresnel integrals

$$\begin{aligned} C(t) &= \int_0^t \cos\tfrac{\pi u^2}{2}du \\ S(t) &= \int_0^t \sin\tfrac{\pi u^2}{2}du \end{aligned} \tag{1.1}$$

Hereby, we get the following parametric representation of clothoids:

$$\mathbf{c}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = b \begin{pmatrix} C(t) \\ S(t) \end{pmatrix} = b \begin{pmatrix} \int_0^t \cos\frac{\pi u^2}{2}du \\ \int_0^t \sin\frac{\pi u^2}{2}du \end{pmatrix} \tag{1.2}$$

with clothoid parameters $b = a\sqrt{\pi}$ or $a = \frac{b}{\sqrt{\pi}}$, resp.
Curvature and arc length of clothoids are proportional

$$\text{curvature } \kappa = \frac{\pi t}{b} = \frac{\sqrt{\pi} t}{a} \tag{1.3}$$

$$\text{arc length } s = bt = at\sqrt{\pi} \tag{1.4}$$

$$\rightarrow \kappa = \frac{1}{a^2} \cdot s \tag{1.5}$$

As $t \rightarrow \pm\infty$, $c(t)$ spirals around the - so called - asymptotic points $P_{asy}^{1,2}$ with values

$$P_{asy}^1 = \left(\frac{a\pi}{2}, \frac{a\pi}{2}\right), P_{asy}^2 = \left(-\frac{a\pi}{2}, -\frac{a\pi}{2}\right) \tag{1.6}$$

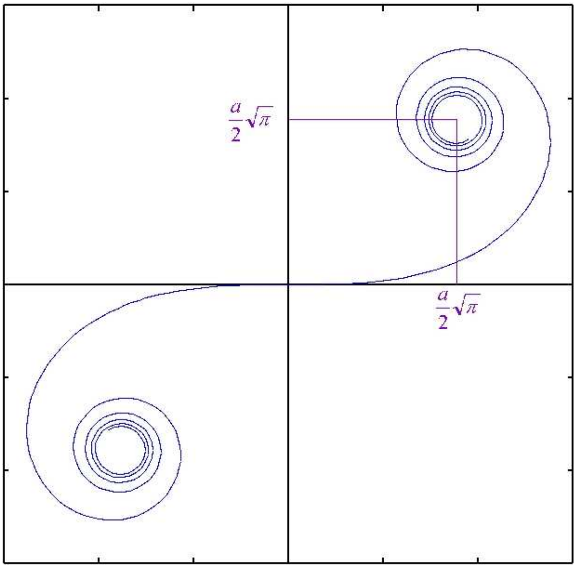

Figure 1: Clothoid

2. Approximation of the clothoid with s-power series

As mentioned above the Fresnel integrals (1.1) cannot be described in an analytic sense. Therefore several approximations are known. We decided to use an approach, which is based on Hermitian splines [1]. This method has been proved to be very suitable for our approximation problem (see [2]).

2.1. Motivation

An approved method for putting together piecewise defined functions

$$f : [t_0, t_1] \rightarrow \mathbb{R}$$

is the use of Two-Point-Hermite-Interpolants $H_k(f;t)$ which reproduce the k-th derivative at the endpoints of $f(t)$. This polynomial converges against $f(t)$ if the function does not have singularities in the complex plane, which is true for the Fresnel integrals. Furthermore we get C^k-continuity at the endpoints if $H_k(f;t)$ has degree $2k+1$. Unfortunately polynomials of higher degree can not be derived from those of lower degree by just adding higher-order terms as this is possible e.g. with Taylor polynomials. To evade this problem [1] introduces a new basis – the symmetric counterpart of power series.

2.2. Definition of s-power series

A s-power series over the unit interval $[0,1]$ is a power series with symmetric parameter $s=(1-u)u\ u\in[0,1]$, whose coefficients a_k are linear functions in u:

$$a(u)=\sum_{k=0}^{\infty} a_k(u)s^k \quad \text{with} \quad \begin{pmatrix} s=(1-u)u \\ a_k(u)=(1-u)a_k^0+ua_k^1 \end{pmatrix} \tag{2.1}$$

Observe, that a_k is already given in the Bernstein basis. s^k can be interpreted as the k-th scaled central Bernstein polynomial:

$$s^k(u)=\begin{pmatrix} 2k \\ k \end{pmatrix}^{-1} B_k^{2k}(u) \quad \text{with} \quad B_k^{2k}(u)=\begin{pmatrix} 2k \\ k \end{pmatrix}(1-u)^k u^k \tag{2.2}$$

This property proves to be very useful if we want to calculate the Bézier representation of the approximating curve. A function $a(u)$ can be expressed as a unique s-power series $H_k(a;u)=\sum_{i=0}^{k} a_i(u)s^i$.

2.3. Computing s-power series of the clothoid and its offset

[1] explains, how to compute the coefficients a_k^0 and a_k^1 (2.1) for the Fresnel integrals step-by-step based on the calculation of s-power series for fundamental operations like addition, multiplication, composition ...; as well as for trigonometric functions and integrals. By combining them, nearly every function can be described as s-power series.

Approximating the offset of a clothoid $\mathbf{c}_d(t)$ for a displacement d is quite easy, because the offset has the form

$$\mathbf{c}_d(t)=\mathbf{c}(t)\pm d\mathbf{n}(t), \quad \mathbf{n}(t)=\begin{pmatrix} -\sin\frac{\pi}{2}t^2 \\ \cos\frac{\pi}{2}t^2 \end{pmatrix} \tag{2.3}$$

So we just have to compute – in addition to the approximation of the clothoid $\mathbf{c}(t)$ – the s-power series for the sinusoidal functions contained in the unit normal $\mathbf{n}(t)$ (2.3).

2.4. Bézier representation of the approximating curve

An elegant and numerically stable way of computing the Bézier representation of an approximation based on s-power series is to utilize the form of the symmetric parameter s (2.1). A function in s-power representation

$$a(u) = \sum_k a_k(u)s^k$$

can be expressed as:

$$a(u) = \sum_k (a_k^0(1-u) + a_k^1 u)(1-u)^k u^k = \sum_k [a_k^0(1-u)^{k+1}u^k + a_k^1(1-u)^k u^{k+1}]$$

or with Bernstein polynomials:

$$a(u) = \sum_k [\hat{a}_k^0 B_k^{2k+1}(u) + \hat{a}_k^1 B_{k+1}^{2k+1}(u)] \tag{2.4}$$

where

$$\hat{a}_k^0 = \begin{pmatrix} 2k+1 \\ k \end{pmatrix}^{-1} a_k^0, \quad \hat{a}_k^1 = \begin{pmatrix} 2k+1 \\ k+1 \end{pmatrix}^{-1} a_k^1.$$

Every "summand" consists of two Bernstein polynomials $B_k^{2k+1}(u)$ and $B_{k+1}^{2k+1}(u)$. Let $H_l(a;u)$ be a Hermite approximation. We want to have the Bézier representation with Bernstein polynomials of degree $2l+1$. Therefore we just need repeated degree elevation up to the degree $2l+1$ applied to every "summand". Corresponding coefficients can be combined and we get the Bézier representation of $H_l(a;u)$.

3. Approach for 3D clothoids

3.1. 3D-clothoids

It was mentioned in the abstract, that clothoids are used especially for road design as manifolds between straight lines and arcs of circles. Since roads are not always plane, it would be nice, to have a three dimensional representative of clothoids. But clothoids are only defined in 2D-space, thus we get the general representation

$$\mathbf{c}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = b \begin{pmatrix} \int_0^t \cos\frac{\pi u^2}{2} du \\ \int_0^t \sin\frac{\pi u^2}{2} du \\ f(t) \end{pmatrix} 216.96 \tag{3.1}$$

with some function $f(t)$. (In the following, we set $b = 1$.)

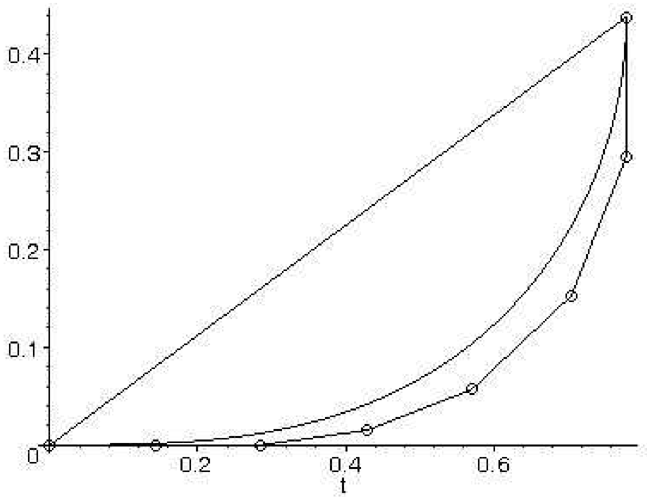

Figure 2: order-3 Hermite approximation of the clothoid ($k = 3$, degree $n = 7$) together with its corresponding Bézier-polygon, domain $t \in [0, 1]$

3.2. Linear approach

As we can choose any function type, the easiest way for defining $f(t)$ is a linear approach

$$z(t) = f(t) = kt \tag{3.2}$$

with $k = const$, $k \in \mathbb{R}$.
Geometrical properties:

$$\text{curvature } \kappa = \frac{\pi}{1 + k^2} \cdot t \tag{3.3}$$

$$\text{arc length } s = \sqrt{1 + k^2} \cdot t \tag{3.4}$$

$$\text{torsion } \tau = \frac{k\pi}{1 + k^2} \cdot t \tag{3.5}$$

From (3.3) and (3.4) follows

$$\kappa \sim s \tag{3.6}$$

so the main property of 2D-clothoids also holds for 3D-clothoids. (3.3) and (3.5) provides

$$\frac{\tau}{\kappa} = k = const \tag{3.7}$$

Since the fraction τ/κ is constant, 3D-clothoids with linear z-functions are cylindrical helixes.

4. Conclusions and outlook

We described an approximation method for a clothoid and its offset respectively, which is very accurate and simultaneously uses polynomials of low degree. Based on that technique we can approximate the 3D-clothoid treated in section 3. Investigations have shown, that Hermitian splines do a good job for the 3D case as well. It can be expected, that the proposed approximation yields good results for a generalized height function $f(t)$ (3.1). Our focus laid on the proportionality of arc length and curvature of the approximating curve. Further research on planes which include the clothoid and its offsets (in the 3D case) has started. Therefore we need to know more about properties of the desribed surface, such as transversal slope, certain curvatures, etc. It is also interesting to study geometrical attributes of offset curves of the clothoid.

References

[1] Sánchez-Reyes, J., Chacón, J. M., Polynomial approximation to clothoids via s-power series, *Computer Aided Design*, 35, (2003), 1305–13.

[2] Weidner, T., Approximation von Klotoiden, *Diplomarbeit im Diplomstudiengang Mathematik an der Fachhochschule Stuttgart – Hochschule für Technik*, (2004).

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