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On some nonstandard extensions of Heyting Arithmetic

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Abstract

In the first part of the paper Heyting Arithmetic augmented with axioms $\underline{n} < c$ for a new constant c is examined from a constructive point of view, namely its consistency with a version of Church's Thesis is proved. In the second part a different extension of HA is treated, an expected property is proved for a new relation expressing feasibility in arithmetic.

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The following examinations are concerned with certain extensions of the so-called Heyting Arithmetic (HA), which is the usual theory of arithmetic furnished with the intuitionistic predicate calculus. In relation to the first of these theories, which will be denoted as HA^c, we shall discuss a question which can be considered as specific to constructive interpretations of mathematical theories. Concerning our second theory, we are going to present a simple proof theoretic method to understand the meaning of a certain predicate symbol more thoroughly. Prior to all these let us give a detailed description of HA^c.

Our logic is intuitionistic predicate logic, which is the usual predicate calculus omitting the rule for double negation.

The language is the usual one-sorted language of arithmetic with possibly indexed variables denoted by lower case letters $x, y, z...$ and with constant symbols $0, c$.

We assume that our language of arithmetic contains function symbols for all primitive recursive functions. Our theory is amplified with the defining equations for all primitive recursive functions. Thus, we understand by HA an extension of the usual theory of arithmetic, which is a conservative extension of it [3].

As a matter of convenience we adopt $<$, besides $=$, as a predicate symbol of arithmetic together with the axioms:

1. $\forall xyz(x < y \wedge y < z \supset x > z)$,
2. $\forall x \neg (x < x)$,
3. $\forall xy(x < y \vee x = y \vee y < x)$,
4. $0 < 1 \wedge \forall x(x > 0 \supset x = 1 \vee x > 1)$,
5. $\forall x(x > 0 \vee x = 0)$.

Let us extend HA with axioms $\underline{n} < c$ for all natural numbers n. The resulting theory will be termed as HA^c.

As early as 1934 Skolem proved the existence of models of nonstandard classical, that is Peano, arithmetic (PA). In fact, Skolem proved with a simple argument based on the compactness theorem that the set

$$Th(\mathbb{N}) \cup \{\underline{n} < c | n \in \mathbb{N}\}$$

is consistent, where

$$Th(\mathbb{N}) = \{\sigma | \sigma \text{ is a sentence of PA}, \mathbb{N} \models \sigma\}.$$

The following proposition provides a similar result but from a proof-theoretic point of view, and it also gives some insight into the main thread of the subsequent arguments.

Proposition 1: If $HA^c \vdash A(c)$, then there exists a natural number n, for which $HA \vdash (\forall x > n)A(x)$.

Proof: When proving $A(c)$ in HA^c we only make use of a finite number of the axioms $\underline{n} < c$, so in the derivation we can replace c by a new variable supposed to be greater than the maximum of the values occurring in the axioms $\underline{n} < c$ applied in the proof.

Corollary: If HA is consistent, then HA^c is also consistent.

Proof: Take $0 = 1$ to be A in the previous lemma.

Intuitionistic logic has a different flavour to classical logic, for example- making use of the method of realizability devised by Kleene- it can be shown that Heyting Arithmetic (HA) can be extended with an axiom scheme expressing constructivity, which is usually termed as Church's Thesis

$$(CT) \qquad \forall x \exists y A(x, y) \supset \exists e \forall x \exists y (T(e, x, y) \wedge A(x, Uy)),$$

such that the resulting theory remains consistent just in case HA is consistent. It should be remarked that PA is inconsistent with CT [1].

The question arises naturally, whether HA^c could be extended with CT preserving consistency relative to the original theory. Instead of CT we are going to consider a more general scheme, termed as ECT, which is an implicative form of CT with an almost negative formula in the antecedent:

$$(ECT) \qquad \forall x(B(x) \supset \exists y A(x,y)) \supset \exists e \forall x(B(x) \supset \exists y(T(e,x,y) \wedge A(x,Uy))),$$

where B is almost negative.

Remember that a formula is almost negative, if it does not contain disjunction and existential quantification occurs in it in front of atomic subformulas, only. The axiom scheme ECT is also consistent with HA [1]. In our case the consistency result can be obtained easily making use of the analogous result for $HA + ECT$. We can state it as follows.

Proposition 2: If HA is consistent, then $HA^c + ECT$ is also consistent.

Proof: By the argument used in Proposition 1, for every formula $A(c)$, such that $HA^c + ECT \vdash A(c)$, there exists a natural number n, for which $HA + ECT \vdash (\forall y > \underline{n})A(y)$. So the relation $HA^c + ECT \vdash 0 = 1$ would imply $HA + ECT \vdash 0 = 1$, which cannot hold, if HA is consistent.

The next part of our exposition differs in its approach a little bit from the one treated so far.

We are going to consider a theory obtained from HA^c by extending it further and also making some modifications in it. Let F be a one-place predicate symbol. Add to HA^c the following axioms in relation to F:

1. $F(0)$
2. $\forall x \forall y(F(x) \wedge y < x \supset F(y))$
3. $\forall x(F(x) \supset x < c)$
4. $\forall x_1 ... \forall x_n(F(x_1) \wedge ... \wedge F(x_n) \supset F(g(x_1, ..., x_n)))$, for each symbol g standing for a primitive recursive function
5. $A(0) \wedge \forall^f x(A(x) \supset A(Sx)) \supset \forall^f x A(x)$, where $A(x)$ does not contain F and $\forall^f x A(x)$ means $\forall x(F(x) \supset A(x))$.

Finally, we shall omit the usual induction scheme of HA^c, so only the induction scheme expressed by the last item of the above definition remains. The new theory will be referred to as HAF^f. (The superscript f alludes to the presence of an induction scheme with restricted quantifiers.)

We are going to prove a statement asserting an expected property of the predicate F.

Theorem 1: Let us suppose $HAF^f \vdash F(t)$ for some term t. Then there exists a natural number n, such that $HAF^f \vdash t = \underline{n}$.

Prior to proving the theorem, we would introduce some basic notation concerning the theory of recursive functions.

Let $T_n(e, x_1, ..., x_n, z)$ and $U(z)$ denote the Kleene-predicate and the result extracting primitive recursive function usually associated with it, respectively. Recall that $T_n(e, x_1, ..., x_n, z)$ is a primitive recursive relation, which holds if e is the Gödel-number of a computation with arguments $x_1, ..., x_n$ and result $U(z)$, z coding the whole computation process.

We are going to apply the following notations frequently

$$\{e\}(m_1, ..., m_n) = k \leftrightharpoons \exists z(T_n(e, m_1, ..., m_n, z) \wedge U(z) = k),$$

$$!\{e\}(m_1, ..., m_n) \leftrightharpoons \exists y\{e\}(m_1, ..., m_n = y),$$

$e, m_1, ..., m_n, k$ denoting natural numbers.

In the definition below we shall make use of the following abbreviation:

$$!\{n\}(m) \wedge \{n\}(m)rA \leftrightharpoons \exists v(\{n\}(m) = v) \wedge \forall v(\{n\}(m) = v \supset vrA),$$

where A is any formula of HAF^f and n, m are natural numbers.

Without proof we shall state the following assertion:

Proposition: There exists a general recursive function $h : \mathbb{N} \to \mathbb{N}$ which enumerates the Gödel-numbers of the terms of HAF^f (not necessarily injectively).

Let us define a realizability relation as follows.

Definition: We shall define realizability for atomic formulas first.

1. $$nrt = s \leftrightharpoons Prf(n, \ulcorner t = s \urcorner),$$

   where $n \in \mathbb{N}$, Prf is a primitive recursive relation coding provability in HAF^f.

2. The case of the formulas of the form $t < s$ is quite similar to the previous one.

3. $nrF(t) \leftrightharpoons j_1(n)rt = \underline{j_2(n)}$, where t is a term of HAF^f.

Definition:

1. $nrA \wedge B \leftrightharpoons j_1(n)rA \wedge j_2(n)rB$

2. $nrA \vee B \leftrightharpoons j_1(n) = 0 \supset j_1(j_2)(n)rA \wedge j_1(n) \neq 0 \supset j_2(j_2)(n)rB$
3. $nrA \supset B \leftrightharpoons \forall m(mrA \supset !\{n\}(m) \wedge \{n\}(m)rB)$
4. $nr\forall xA(x) \leftrightharpoons \forall m(!\{n\}(m) \wedge \{n\}(m)rA(\overline{h(m)}))$, where $\overline{h(m)}$ is the term of which the Gödel-number is $h(m)$.
5. $nr\exists xA(x) \leftrightharpoons j_1(n)rA(\overline{h(j_2(n))})$

Theorem 2: If $HAF^f \vdash A$, then there is a natural number n such that nrA holds.

Proof: The proof goes along the lines of the proof of the standard argument for the original Kleene-type realizability [1], some care is needed only when realizing the induction axiom and the axioms in connection with F.

Let us treat the induction axiom first. Let us suppose $nrA(0) \wedge \forall^f x(A(x) \supset A(Sx))$. We have to find a partial recursive function which, applied to n, supplies a value realizing $\forall^f xA(x)$. To this end let $wr\forall^f xA(x)$ and $vnF(\overline{h(k)})$ be valid for some natural numbers w, v and k. By definition $vnF(\overline{h(k)}) \leftrightharpoons Prf(j_1(v), \ulcorner \underline{j_2(v)} = \overline{h(k)} \urcorner)$, which implies, $j_1(v)$ being a natural number, the provability of $\underline{j_2(v)} = \overline{h(k)}$ in HAF^f. From the lemma below we can conclude that it is enough to construct a number for the realizability of $A(\underline{j_2(v)})$.

First of all $j_1(n)rA(0)$. We can find a number n_0, such that $h(n_0) = \ulcorner 0 \urcorner$. Let us suppose that m_0 is a number realizing $F(0)$.
Then $\{\{\{j_2(n)\}(n_0)\}(m_0)\}(j_1(n))rA(S0)$.

Find numbers n_1, m_1 now, such that $h(n_1) = \ulcorner S0 \urcorner$ and $m_1 rF(S0)$. Then, denoting the term above obtained for $A(S0)$ by s_0, we can write
$\{\{\{j_2(n)\}(n_1)\}(m_1)\}(s_0)rA(SS0)$, thus we get an s_1 realizing $A(SS0)$.

Continue this process until we find an $s_{j_2(v)}$ which realizes $A(\underline{j_2(v)})$.

Let us examine one more axiom, namely $\forall x(F(x) \supset x < c)$. Assume for natural numbers n and m we have $nrF(\overline{h(m)})$. Then $Prf(j_1(n), \ulcorner \underline{j_2(n)} = \overline{h(m)} \urcorner)$, that is, given a Gödel-number of the proof of $\underline{j_2(n)} = \overline{h(m)}$, we can calculate using n and m a Gödel-number for the proof of $\overline{h(m)} < c$, from which we can deduce that the realizability of the underlying axiom holds.

Lemma: For every formula A of HA^c there is a natural number n realizing $\forall x \forall y(x = y \supset (A(x) \supset A(y)))$.

Proof: The proof is by induction with respect to the logical complexity of A.

Now we can turn to the proof of Theorem 1.

Proof of Theorem 1: Let us suppose $HAF^f \vdash F(t)$ for some term t. Then, by the above theorem, there is an n for which $nrF(t)$ holds. This means $Prf(j_1(n), \ulcorner \underline{j_2(n)} = t \urcorner)$, by which the assertion of the theorem follows.

References

[1] A. G. Dragalin: *Mathematical Intuitionism. Introduction to Proof Theory.*, Translations of AMS, 67., 1988.

[2] A. S. Troelstra, D. van Dalen: *Constructivism in Mathematics*, Vol. I.-II., North-Holland, 1988.

[3] S. C. Kleene: *Introduction to Metamathematics*, North Holand, Amsterdam, 1952.