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A polynomial interpolation

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Abstract

The polynomial interpolation method is an approximation of function represented by its base point. It is assumed that the searched function is in polynomial form.

This paper presents a method of polynomial interpolation where the n^{th} degree approximation of the searched function is a projection on the subspace, which is generated by the functions $1, x, x^2, \ldots, x^n$.

We introduce the orthogonal polynomials system and the scalar product of the functions. The searched polynome is given by projecting the searched function on the orthogonal polynome system.

The method above assumed that the x values of the basic points are fitted in the range [0,1]. We assumed moreover, that the scalar product integrals could be calculated precisely.

Using a shrinking, we can satisfy the first assumption: transforming the basic points x values on the [0,1] range, then use the inverse transformation on the result a_i values. The second assumption also can be satisfied by the error correction transformation, which will be explained later.

The theory of the interpolation

Let given (x_i, y_i) $(i = 0, 1, \ldots n)$ pairs. We assume that y_i $(i = 0, 1, \ldots n)$ are values of the $f(x)$ function in the x_i $(i = 0, 1, \ldots n)$ different places. We name these (x_i, y_i) datum pairs as the base points of the interpolation.
We assume furthermore, that y can be scribed in the following form:

$$f(x) = \sum_{j=0}^{\infty} k_j x^j \tag{1}$$

Let y a function over x, which can be written in the form:

$$y(x) = \sum_{j=0}^{m} a_j x^j \tag{2}$$

We assume on y, that it is an approximation to f, where $m < n$.

We seek the $y(x)$ function by the form (2). Using matrix representation, (2) can be written as

$$y = AX \tag{3}$$

where:

$$A = (a_0, a_1, .., a_m) \tag{4}$$

$$X = \begin{pmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^n \end{pmatrix} \tag{5}$$

Let S_x the space of the linear combination of the functions $(1, x, x^2, \ldots, x^m, \ldots)$. Every $y \in S_x$ function can be transformed into the form

$$y = \sum_{j=0}^{m} b_j e_j \tag{6}$$

that is in the scheme of (3), (4) and (5):

$$y = BE \tag{7}$$

We take a constraint on E as:

$$\langle e_i \mid e_j \rangle = \delta_{ij} \tag{8}$$

Where the definition of the scalar product is:

$$\langle f \mid h \rangle = \int_0^1 fh dx \tag{9}$$

if f and h were functions over x.
We take another constraint on E:

$$\left\langle x^k \mid e_j \right\rangle = 0 \qquad \text{if } j > k \tag{10}$$

That is, y is the orthogonal projection of f on the subspace $(1, x, x^2, \dots, x^m)$. (See (1)).
Thus, it is true on y, that

$$b_i = \langle y \,| e_i \rangle \tag{11}$$

In the case, that there exists the matrix

$$C = \begin{pmatrix} c_{00} & c_{01} & c_{02} & c_{0m} \\ c_{10} & c_{11} & c_{12} & c_{1m} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m0} & c_{m1} & c_{m2} & c_{mm} \end{pmatrix} \tag{12}$$

that

$$E = CX \tag{13}$$

then

$$y = BE = AX \tag{14}$$

that is

$$A = BC \tag{15}$$

using (11) and (15) A, that is y can be determined.

Transformation of the base points

Regarding on (9), we see that the interval of the scalar product is *[0,1]*.
Introducing the variable t, where

$$t_i = \frac{x_i - x_0}{x_n - x_0} \tag{16}$$

the base points are transformed into *[0,1]*.
It is easy to prove, that using the

$$c_{ij} = \sqrt{2i+1}(-1)^j \begin{pmatrix} i \\ j \end{pmatrix} \begin{pmatrix} i+j \\ j \end{pmatrix} \tag{17}$$

coefficients, E satisfies the orthogonal criteria (13).

B can be calculated by using numeric integrals in (11).
Considering (15), we can say the $y(t)$ is determined Then, as it is shown in (14) and (15)

$$y = BE = HT \tag{18}$$

where

$$T = \begin{pmatrix} 1 \\ t \\ t^2 \\ \vdots \\ t^m \end{pmatrix} \tag{19}$$

Because

$$T = QX \tag{20}$$

where

$$q_{ij} = \begin{pmatrix} j \\ i \end{pmatrix} \frac{1}{(x_n - x_0)^i} (-x_0)^{(j-i)} \tag{21}$$

(18) can be transformed as

$$y = BE = HT = (BC)T = BCQX \tag{22}$$

that is

$$A = BCQ \tag{23}$$

That means that we got the form of y.

The miscalculation of the numeric integral

The miscalculation of the (11) numeric integrals at determining B, is the error of the interpolation method. In the followings, we give a method how to correct the miscalculation.

Let

$$I(f, g) \tag{24}$$

Is the numeric integral of the function f and g, for example, by the trapeze formula, on $[1, 0]$, where f and g are function of x.
Generally true at calculation of b_i, that

$$d_i = I(y, e_i) <> \langle y \mid e_i \rangle = b_i \tag{25}$$

It is true in (24) numeric integral, that

$$I(y, e_i) = I(\sum_{j=0}^{m} b_j e_j, e_i) = \sum_{j=0}^{m} b_j I(e_j, e_i) \tag{26}$$

Let

$$n_{ji} = I(e_j, e_i) \tag{27}$$

This way, (25), (26) and (27) can be written in the form

$$D = BN \tag{28}$$

where D is a row matrix of the d_i numeric integrals, B is a row matrix of the b_i coefficients, while N is the matrix of n_{ij}'s

In a case, if $i, j < n$, that is the dimension of N is less the the number of the base points, there exists the inverse of N.
Using N^{-1} it can be written as

$$B = DN^{-1} \tag{29}$$

This and (23) say that

$$A = ((DN^{-1})C)Q \tag{30}$$

That is the miscalculation of the trapeze (or any other) numeric integral has been corrected.
This means moreover, that in the case of $m = n - 1$, the corrected coefficients give the m-degree Lagrange-polynome.