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The Asymptotic Covariance of Kernel Type Density Estimators for Random Fields*

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Abstract

Kernel type density estimators are studied for random fields. It is proved that the estimators are asymptotically normal if the set of locations of observations become more and more dense in an increasing sequence of domains. It turns out that in our setting the covariance structure of the limiting normal distribution can be a combination of those of the continuous parameter and the discrete parameter cases. The proof is based on a new central limit theorem for α-mixing random fields. Simulation results support our theorems.

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1. Introduction

The main result of this paper is Theorem 2.1. It states asymptotic normality of the kernel type density estimator when the set of locations of observations become more and more dense in an increasing sequence of domains. It turns out, that the covariance structure of the limit normal distribution depends on the ratio of the bandwidth of the kernel estimator and the diameter of the subdivision. This is an important issue when we approximate the integral in the estimator $f_{T_n}(x) = \frac{1}{|T_n|}\frac{1}{h_n}\int_{T_n} K\left(\frac{x-\xi_{\mathbf{t}}}{h_n}\right) d\mathbf{t}$ by a sum, i.e. in practical applications we use an estimator of the form $f_{\mathcal{D}_n}(x) = \frac{1}{|\mathcal{D}_n|}\frac{1}{h_n}\sum_{\mathbf{i}\in\mathcal{D}_n} K\left(\frac{x-\xi_{\mathbf{i}}}{h_n}\right)$.

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Kernel type density estimators are widely studied, see e.g. Bosq [1], Kutoyants [10], Prakasa Rao [14]. Several papers are devoted to the density estimators for weakly dependent stationary sequences (see, e.g., Castellana and Leadbetter [3], Bosq et al. [2], Liebscher [13]). A few papers study the relation of the rate of dependence and the asymptotic behaviour (see, e.g., Csörgő and Mielniczuk [5]).

The asymptotic normality of the kernel type density estimator is well known for weakly depenedent continuous time processes (see, e.g., Bosq et al. [2]). However, when we calculate numerically the kernel type density estimator, its asymptotic variance can be different from that of the theoretical one. To point out this phenomenon is the goal of our paper, and therefore we turn to so called infill-increasing setup.

In statistics, most asymptotic results concern the increasing domain case, i.e. when the random process (or field) is observed in an increasing sequence of domains T_n, with $|T_n| \to \infty$. However, if we observe a random field in a fixed domain and intend to prove an asymptotic theorem when the observations become dense in that domain, we obtain the so called infill asymptotics (see Cressie [4]). In this paper we combine the infill and the increasing domain approaches. We call infill-increasing approach if our observations become more and more dense in an increasing sequence of domains. Using this setup, Lahiri [11] and Fazekas [7] studied the asymptotic behaviour of the empirical distribution function. General central limit theorems were obtained in Lahiri [12] for spatial processes under infill-increasing type designs.

The main result is in Section 2. Theorem 2.1 states asymptotic normality of the kernel type density estimator (3) in the infill-increasing case. The underlying random field is α-mixing. The conditions are similar to those of Theorem 2.2 (continuous time process) and Theorem 3.1 (discrete time process) of Bosq et al. [2]. Our result is in some sense between the discrete and the continuous time cases. The proof is given in Section 3. The basic tool is Theorem 3.1. It is a central limit theorem for random fields analogous to Theorem 1.1 in Bosq et al. [2]. In Section 4 we give examples. Simulation results support the covariance structure of the limit distribution presented in Theorem 2.1. The results of this paper were announced at conferences, see e.g. Fazekas [8].

2. Asymptotic normality of density estimators

The following notation is used. $\mathbb{N}$ is the set of positive integers, $\mathbb{Z}$ is the set of all integers, $\mathbb{N}^d$ and $\mathbb{Z}^d$ are d-dimensional lattice points, where d is a fixed positive integer. $\mathbb{R}$ is the real line, $\mathbb{R}^d$ is the d-dimensional space with the usual Euclidean norm $\|\mathbf{x}\|$. In $\mathbb{R}^d$ we shall also consider the distance corresponding to the maximum norm: $\varrho(\mathbf{x}, \mathbf{y}) = \max_{1\le i\le d} |x^{(i)} - y^{(i)}|$, where $\mathbf{x} = (x^{(1)}, \dots, x^{(d)})$, $\mathbf{y} = (y^{(1)}, \dots, y^{(d)})$. The distance of two sets in $\mathbb{R}^d$ corresponding to the maximum norm is also denoted by ϱ: $\varrho(A, B) = \min\{\varrho(a, b) \,:\, a \in A,\, b \in B\}$.

$|\mathcal{D}|$ denotes the cardinality of the finite set $\mathcal{D}$ and at the same time $|T|$ denotes the volume of the domain T.

We suppose the existence of an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The σ-

algebra generated by a set of random variables is denoted by $\sigma\{.\}$. Sign $\mathbb{E}$ stands for the expectation. The variance and the covariance are denoted by var(.) and cov(.,.), respectively.

Sign $\Rightarrow$ denotes convergence in distribution. $\mathcal{N}(m, \Sigma)$ stands for the (vector) normal distribution with mean (vector) m and covariance (matrix) Σ.

Describe the scheme of observations. Let $\Lambda > 0$ be fixed. By $(\mathbb{Z}/\Lambda)^d$ we denote the Λ-lattice points in $\mathbb{R}^d$ i.e. lattice points with distance $1/\Lambda$:

$$\left(\frac{\mathbb{Z}}{\Lambda}\right)^d = \left\{ \left(\frac{k_1}{\Lambda}, \dots, \frac{k_d}{\Lambda}\right) \ : \ (k_1, \dots, k_d) \in \mathbb{Z}^d \right\}.$$

Let $T_1, T_2, \dots$ be bounded, closed rectangles in $\mathbb{R}^d$ with edges parallel to the axes. Suppose that

$$T_1 \subset T_2 \subset T_3 \subset \dots, \quad \bigcup\nolimits_{i=1}^{\infty} T_i = T_\infty. \tag{1}$$

We assume that the length of each edge of T_n is integer and converges to ∞, as $n \to \infty$. Let $\{\Lambda_n\}$ be an increasing sequence of positive integers and let $\mathcal{D}_n$ be the Λ_n-lattice points belonging to T_n, i.e., $\mathcal{D}_n = T_n \cap (\mathbb{Z}/\Lambda_n)^d$.

Let $\xi_{\mathbf{t}}$, $\mathbf{t} \in T_\infty$, be a strictly stationary random field with unknown continuous marginal density function f. The n-th set of observations involves the values of the random field $\xi_{\mathbf{t}}$ taken at each point $\mathbf{k} \in \mathcal{D}_n$. We shall estimate f from the data $\xi_{\mathbf{k}}$, $\mathbf{k} \in \mathcal{D}_n$. Actually, each $\mathbf{k} = \mathbf{k}^{(n)} \in \mathcal{D}_n$ depends on n but to avoid complicated notation we often omit superscript (n). By our assumptions, $\lim_{n\to\infty} |\mathcal{D}_n| = \infty$.

We need the notion of α-mixing (see e.g. Doukhan [6]). Let $\mathcal{A}$ and $\mathcal{B}$ be two σ-algebras in $\mathcal{F}$. The α-mixing coefficient of $\mathcal{A}$ and $\mathcal{B}$ is defined as follows.

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup\{|\mathbb{P}(A)\mathbb{P}(B) - \mathbb{P}(AB)| \ : \ A \in \mathcal{A}, B \in \mathcal{B}\}.$$

The α-mixing coefficient of $\{\xi_{\mathbf{t}} \ : \ \mathbf{t} \in T_\infty\}$ is

$$\alpha(r) = \sup\{\alpha(\mathcal{F}_{I_1}, \mathcal{F}_{I_2}) \ : \ \varrho(I_1, I_2) \geq r\},$$

where I_i is a finite subset in T_∞ and $\mathcal{F}_{I_i} = \sigma\{\xi_{\mathbf{t}} \ : \ \mathbf{t} \in I_i\}$, $i = 1, 2$. We shall use the following condition. For some $1 < a < \infty$

$$\int_0^\infty s^{2d-1} \alpha^{\frac{a-1}{a}}(s) ds < \infty . \tag{2}$$

A function $K : \mathbb{R} \to \mathbb{R}$ will be called a kernel if K is a bounded, continuous, symmetric density function (with respect to the Lebesgue measure), and $\lim_{|u|\to\infty} |u|K(u) = 0$, $\int_{-\infty}^{+\infty} u^2 K(u)\, du < \infty$. Let K be a kernel and let $h_n > 0$, then the kernel-type density estimator is

$$f_n(x) = \frac{1}{|\mathcal{D}_n|} \frac{1}{h_n} \sum\nolimits_{\mathbf{i} \in \mathcal{D}_n} K\left(\frac{x - \xi_{\mathbf{i}}}{h_n}\right), \qquad x \in \mathbb{R}. \tag{3}$$

Let $f_{\mathbf{u}}(x, y)$ be the joint density function of $\xi_{\mathbf{0}}$ and $\xi_{\mathbf{u}}$, $\mathbf{u} \neq \mathbf{0}$. Denote $\mathbb{R}^d_{\mathbf{0}}$ the set $\mathbb{R}^d \setminus \{\mathbf{0}\}$. Let

$$g_{\mathbf{u}}(x, y) = f_{\mathbf{u}}(x, y) - f(x)f(y), \quad \mathbf{u} \in \mathbb{R}^d_{\mathbf{0}}, \ x, y \in \mathbb{R}. \tag{4}$$

We assume that $g_{\mathbf{u}}(x,y)$ is continuous in x and y for each fixed $\mathbf{u}$. Let $g_{\mathbf{u}}$ denote $g_{\mathbf{u}}(x,y)$ as a function $g:\mathbb{R}^d_{\mathbf{0}} \to C(\mathbb{R}^2)$, i.e. a function with values in $C(\mathbb{R}^2)$, the space of continuous real-valued functions over $\mathbb{R}^2$. Let $\|g_{\mathbf{u}}\| = \sup_{(x,y)\in\mathbb{R}^2} |g_{\mathbf{u}}(x,y)|$ be the norm of $g_{\mathbf{u}}$.

For a fixed positive integer m and fixed distinct real numbers $x_1,\dots,x_m$, let

$$\sigma(x_i,x_j) = \int_{\mathbb{R}^d_{\mathbf{0}}} g_{\mathbf{u}}(x_i,x_j)\,d\mathbf{u}\,, \quad i,j=1,\dots,m, \tag{5}$$

$$\Sigma^{(m)} = \big(\sigma(x_i,x_j)\big)_{1\le i,j\le m}\,. \tag{6}$$

Theorem 2.1. *Assume that $g_{\mathbf{u}}$ is Riemann integrable (as a function $g:\mathbb{R}^d_{\mathbf{0}} \to C(\mathbb{R}^2)$) on each bounded closed d-dimensional rectangle $R\subset\mathbb{R}^d_{\mathbf{0}}$, moreover $\|g_{\mathbf{u}}\|$ is directly Riemann integrable (as a function $\|g\|:\mathbb{R}^d_{\mathbf{0}}\to\mathbb{R}$). Let $x_1,\dots,x_m$ be given distinct real numbers and assume that $\Sigma^{(m)}$ in (6) is positive definite. Suppose that there exists $1<a<\infty$ such that (2) is satisfied and*

$$(h_n)^{-1} \le c|T_n|^{\frac{a^2}{(3a-1)(2a-1)}} \quad \textit{for each} \quad n\,. \tag{7}$$

Assume that $\lim_{n\to\infty}\Lambda_n=\infty$ and $\lim_{n\to\infty}h_n=0$. If

$$\lim_{n\to\infty}\frac{1}{\Lambda_n^d}\frac{1}{h_n}=0, \tag{8}$$

then

$$\sqrt{\frac{|\mathcal{D}_n|}{\Lambda_n^d}}\Big\{\big(f_n(x_i)-\mathbb{E}f_n(x_i)\big)\,,\, i=1,\dots,m\Big\} \Rightarrow \mathcal{N}(0,\Sigma^{(m)}), \quad \textit{as} \quad n\to\infty. \tag{9}$$

If, instead of (8),

$$\lim_{n\to\infty}\frac{1}{\Lambda_n^d}\frac{1}{h_n}=L>0 \tag{10}$$

is satisfied, then (9) remains valid if $\Sigma^{(m)}$ is changed for

$$\Sigma^{'(m)} = \Sigma^{(m)} + D\,, \tag{11}$$

where D is a diagonal matrix with diagonal elements $Lf(x_i)\int_{-\infty}^{+\infty}K^2(u)\,du$, $i=1,\dots,m$.

If $f(x)$ has bounded second derivative and $\lim_{n\to\infty}|T_n|h_n^4=0$, then in (9) $\mathbb{E}f_n(x_i)$ can be changed for $f(x_i)$, $i=1,\dots,m$, and both of the above statements remain valid.

3. Proof of the main result

First we turn to the version of the central limit theorem appropriate to our sampling scheme. Our Theorem 3.1 is a modification of Theorem 1.1 of Bosq et al. [2]. The novelties of Theorem 3.1 are the infill-increasing setting and that it concerns random fields.

Define the discrete parameter (vector valued) random field $Y_n(\mathbf{k})$ as follows. For each $n = 1, 2, \ldots$, and for each $\mathbf{k} \in \mathcal{D}_n$

$$\text{let } \ Y_n(\mathbf{k}) = Y_n(\mathbf{k^{(n)}}) \ \text{ be a Borel measurable function of } \ \xi_{\mathbf{k}^{(n)}}. \tag{1}$$

Theorem 3.1. *Let $\xi_\mathbf{t}$ be a random field and let $Y_n(\mathbf{k}) = (Y_n^{(1)}(\mathbf{k}), \ldots, Y_n^{(m)}(\mathbf{k}))$ be an m-dimensional random field defined by (1). Let $S_n = \sum_{\mathbf{k}\in\mathcal{D}_n} Y_n(\mathbf{k})$, $n = 1, 2, \ldots$. Suppose that for each fixed n the field $Y_n(\mathbf{k})$, $\mathbf{k} \in \mathcal{D}_n$, is strictly stationary with $\mathbb{E}Y_n(\mathbf{k}) = 0$. Assume that*

$$\|Y_n(\mathbf{k})\| \le M_n, \tag{2}$$

where M_n depends only on n;

$$\sup_{n,\mathbf{k},r} \mathbb{E}\big(Y_n^{(r)}(\mathbf{k})\big)^2 < \infty; \tag{3}$$

for any increasing, unbounded sequence of rectangles G_n with $G_n \subseteq T_n$

$$\lim_{n\to\infty} \frac{1}{\Lambda_n^d |\mathcal{G}_n|} \mathbb{E}\left[\sum_{\mathbf{k}\in\mathcal{G}_n} Y_n^{(r)}(\mathbf{k}) \sum_{\mathbf{l}\in\mathcal{G}_n} Y_n^{(s)}(\mathbf{l})\right] = \sigma_{r,s}, \quad r, s = 1, \ldots, m, \tag{4}$$

where $\mathcal{G}_n = G_n \cap (\mathbb{Z}/\Lambda_n)^d$; the matrix $\Sigma = (\sigma_{r,s})_{r,s=1}^m$ is positive definite; there exists $1 < a < \infty$ such that (2) is satisfied; and

$$M_n \le c|T_n|^{\frac{a^2}{(3a-1)(2a-1)}} \quad \textit{for each} \quad n. \tag{5}$$

Then

$$\frac{1}{\sqrt{\Lambda_n^d |\mathcal{D}_n|}} S_n \Rightarrow \mathcal{N}(0, \Sigma), \qquad \textit{as} \ \ n \to \infty. \tag{6}$$

The proof of Theorem 3.1 can be found in Fazekas and Chuprunov [9].

Remark. We fix the notion of direct Riemann integrability for nonnegative functions defined on $\mathbb{R}^d_\mathbf{0}$ and (possibly) unbounded at the origin.

Let $l : \mathbb{R}^d_\mathbf{0} \to [0, \infty)$ be given. For an $h > 0$ consider a subdivision of $\mathbb{R}^d$ into (right closed and left open) d-dimensional cubes $\Delta_\mathbf{i}$, with edge length h such that the center of $\Delta_\mathbf{0}$ is the origin $\mathbf{0} \in \mathbb{R}^d$. The family $\{\Delta_\mathbf{i}\}$ is called the subdivision corresponding to h. If $\mathbf{i} \ne \mathbf{0}$, for $\mathbf{x} \in \Delta_\mathbf{i}$, then let $\bar{l}_h(\mathbf{x}) = \sup\{l(\mathbf{y}) : \mathbf{y} \in \Delta_\mathbf{i}\}$, $\underline{l}_h(\mathbf{x}) = \inf\{l(\mathbf{y}) : \mathbf{y} \in \Delta_\mathbf{i}\}$, while $\bar{l}_h(\mathbf{x}) = \underline{l}_h(\mathbf{x}) = 0$ if $\mathbf{x} \in \Delta_\mathbf{0}$. If

$$\lim_{h\to 0} \int_{\mathbb{R}^d} \bar{l}_h(\mathbf{x})\, d\mathbf{x} = \lim_{h\to 0} \int_{\mathbb{R}^d} \underline{l}_h(\mathbf{x})\, d\mathbf{x} = I$$

and this common value is finite, then l is called directly Riemann integrable (d.R.i.) and I is its direct Riemann integral.

Using Zorich [15], we obtain the following. Call a stripe a set $M = R_1 \setminus R_2$, where R_1 is a closed d-dimensional rectangle while R_2 ($\emptyset \neq R_2 \subset R_1$) is an open d-dimensional rectangle both having center at the origin. Let $l \geq 0$ be d.R.i. Then l is Riemann integrable on each stripe. The improper integral $\int_{\mathbb{R}^d_{\mathbf{0}}} l(\mathbf{x})\,d\mathbf{x}$ exists and it is equal to the direct Riemann integral of l. Moreover, for any $\varepsilon > 0$ there exists a stripe M such that $\int_{\mathbb{R}^d_{\mathbf{0}}\setminus M} l(\mathbf{x})\,d\mathbf{x} \leq \varepsilon$.

Let h_n be positive numbers converging to zero, and let $\{\Delta^{(n)}_{\mathbf{i}}\}$ be the subdivision corresponding to h_n. Then for any $\varepsilon > 0$ there exists a stripe M such that all Riemannian approximating sums (based on the above subdivisions but not containing term $|\Delta_{\mathbf{0}}| l(\mathbf{x_0})$) of the integral $\int_{\mathbb{R}^d_{\mathbf{0}}\setminus M} l(\mathbf{x})\,d\mathbf{x}$ are less than ε. $\square$

Remark. We shall use the limit relations on p. 36 of Prakasa Rao [14]. In particular, if the density function f is continuous, K is a kernel, then as $h_n \to 0$ ($h_n > 0$), we have the following.

$$\operatorname{var}\Big\{\frac{1}{\sqrt{h_n}} K\Big(\frac{x-\xi_{\mathbf{i}}}{h_n}\Big)\Big\} \to f(x) \int_{-\infty}^{+\infty} K^2(u)\,du\,; \tag{7}$$

$$\operatorname{cov}\Big\{\frac{1}{h_n} K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big)\,,\, \frac{1}{h_n} K\Big(\frac{x_s-\xi_{\mathbf{i}}}{h_n}\Big)\Big\} \to -f(x_r)f(x_s)\,, \quad \text{if } \; x_r \neq x_s. \tag{8}$$

If $f_{\mathbf{u}}(x,y)$ is continuous in x and y, then

$$\operatorname{cov}\Big\{\frac{1}{h_n} K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big)\,,\, \frac{1}{h_n} K\Big(\frac{x_s-\xi_{\mathbf{j}}}{h_n}\Big)\Big\} \to g_{\mathbf{i}-\mathbf{j}}(x_r,x_s)\,, \quad \text{if } \; \mathbf{i} \neq \mathbf{j}. \tag{9}$$

If the second derivative of f is bounded, then

$$\mathbb{E}\frac{1}{h_n} K\Big(\frac{x-\xi_{\mathbf{i}}}{h_n}\Big) - f(x) = \mathrm{O}(h_n^2)\,, \quad \text{as } \; h_n \to 0. \tag{10}$$

This last relation can be proved using Taylor's expansion, see e.g. Bosq [1], p. 44. $\square$

Proof of Theorem 2.1. We have to check the conditions of Theorem 3.1. Let $x_1, \ldots, x_m$ be fixed distinct real numbers and define the m-dimensional random vector $X_n(\mathbf{i})$ with the following coordinates:

$$X_n^{(r)}(\mathbf{i}) = \frac{1}{h_n} K\left(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\right) - \frac{1}{h_n}\mathbb{E}K\left(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\right)\,, \tag{11}$$

for $r = 1, \ldots, m$, and $\mathbf{i} \in \mathcal{D}_n$. Divide T_n into d-dimensional unit cubes (having Λ_n^d points of $\mathcal{D}_n$ in each of them). Denote by $\mathcal{D}'_n$ the set of these cubes. Let $Y_n(\mathbf{k})$ be the arithmetical mean of variables $X_n(\mathbf{i})$ having indices $\mathbf{i}$ in the $\mathbf{k}$-th cube. Then for each fixed n, the field $Y_n(\mathbf{k})$, $\mathbf{k} \in \mathcal{D}'_n$, is strictly stationary with $\mathbb{E}Y_n(\mathbf{k}) = 0$. We shall apply Theorem 3.1 to $Y_n(\mathbf{k})$, $\mathbf{k} \in \mathcal{D}'_n$, i.e. we shall use a non infill form of that theorem.

As

$$\|Y_n(\mathbf{k})\| \leq \frac{2m}{h_n}\|K\|_\infty\,,$$

(7) implies (2) and (5).

To prove (3), we calculate $\mathbb{E}\big(Y_n^{(r)}(\mathbf{k})\big)^2$ such that in the double sum of covariances we separate the variances. We obtain

$$\begin{aligned}\mathbb{E}(Y_n^{(r)}(\mathbf{k}))^2 &= \frac{1}{\Lambda_n^d}\frac{1}{h_n}\mathrm{var}\Big\{\frac{1}{\sqrt{h_n}}K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big)\Big\} \\ &+ \frac{1}{\Lambda_n^{2d}}\sum\sum\nolimits_{\mathbf{i}\neq\mathbf{j}}\mathrm{cov}\Big\{\frac{1}{h_n}K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big),\frac{1}{h_n}K\Big(\frac{x_r-\xi_{\mathbf{j}}}{h_n}\Big)\Big\}.\end{aligned} \tag{12}$$

The bundedness of this expression can be checked similarly to the next part of the proof.

To prove (4), let $\{G_n\}$ be an increasing sequence of d-dimensional rectangles, each G_n being union of d-dimensional unit cubes. Then

$$\frac{1}{|G_n|}\mathbb{E}\left\{\sum_{\mathbf{k}\in G_n\cap\mathbb{Z}^d}Y_n^{(r)}(\mathbf{k})\sum_{\mathbf{l}\in G_n\cap\mathbb{Z}^d}Y_n^{(s)}(\mathbf{l})\right\}= \tag{13}$$

$$=\frac{1}{\Lambda_n^d|\mathcal{G}_n|}\sum_{\mathbf{i}\in\mathcal{G}_n}\sum_{\mathbf{j}\in\mathcal{G}_n}\mathrm{cov}\Big\{\frac{1}{h_n}K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big),\ \frac{1}{h_n}K\Big(\frac{x_s-\xi_{\mathbf{j}}}{h_n}\Big)\Big\}=A+B\,, \tag{14}$$

where $\mathcal{G}_n=G_n\cap(\mathbb{Z}/\Lambda_n)^d$, and A denotes the part of the sum with $\mathbf{i}=\mathbf{j}$, while B denotes the part of the sum with $\mathbf{i}\neq\mathbf{j}$.

For A we have

$$A=\frac{1}{\Lambda_n^d}\mathrm{cov}\Big\{\frac{1}{h_n}K\Big(\frac{x_r-\xi_{\mathbf{i}}}{h_n}\Big),\ \frac{1}{h_n}K\Big(\frac{x_s-\xi_{\mathbf{i}}}{h_n}\Big)\Big\}. \tag{15}$$

If $r\neq s$, using (8), we obtain that $A\to 0$, as $\Lambda_n\to\infty$. However, if $r=s$, by (7),

$$\lim_{n\to\infty}A=Lf(x_r)\int_{-\infty}^{+\infty}K^2(u)du\,,$$

when (10) is satisfied, while $\lim_{n\to\infty}A=0$, when (8) is satisfied. Now, turn to B.

$$\begin{aligned}B=\frac{1}{\Lambda_n^d|\mathcal{G}_n|}\sum\sum\nolimits_{\mathbf{i}\neq\mathbf{j}}\Big\{&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{h_n}K\Big(\frac{x_r-u}{h_n}\Big)\frac{1}{h_n}K\Big(\frac{x_s-v}{h_n}\Big)f_{\mathbf{i}-\mathbf{j}}(u,v)\,dudv- \\ &-\int_{-\infty}^{+\infty}\frac{1}{h_n}K\Big(\frac{x_r-u}{h_n}\Big)f(u)\,du\int_{-\infty}^{+\infty}\frac{1}{h_n}K\Big(\frac{x_s-v}{h_n}\Big)f(v)\,dv\Big\}\,.\end{aligned} \tag{16}$$

As the random field is strictly stationary, we can assume that the center of the rectangle G_n is the origin. Then the set of vectors of the form $\mathbf{i}-\mathbf{j}$ with $\mathbf{i},\mathbf{j}\in\mathcal{G}_n$ is $2\mathcal{G}_n$, where $2\mathcal{G}_n$ is defined as $(2G_n)\cap(\mathbb{Z}/\Lambda_n)^d$. If $\mathbf{u}\in 2\mathcal{G}_n$ is fixed, then denote by $|\mathcal{G}_{n,\mathbf{u}}|$ the number of pairs $(\mathbf{i},\mathbf{j})\in\mathcal{G}_n\times\mathcal{G}_n$ with $\mathbf{i}-\mathbf{j}=\mathbf{u}$. Then

$$B=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left\{\frac{1}{h_n}K\Big(\frac{x_r-u}{h_n}\Big)\frac{1}{h_n}K\Big(\frac{x_s-v}{h_n}\Big)\Big(\frac{1}{\Lambda_n^d}\sum_{\mathbf{u}\in 2\mathcal{G}_n^{\mathbf{0}}}\frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|}g_{\mathbf{u}}(u,v)\Big)\right\}dudv\,, \tag{17}$$

where $2\mathcal{G}_n^{\mathbf{0}} = 2\mathcal{G}_n \setminus \{\mathbf{0}\}$. Now fix an $\varepsilon > 0$. As $\|g_{\mathbf{u}}\|$ is directly Riemann integrable, one can find a stripe $M_\varepsilon \subset \mathbb{R}^d$ (with center in the origin) such that

$$\int_{\mathbb{R}_{\mathbf{0}}^d \setminus M_\varepsilon} \|g_{\mathbf{u}}\| \, d\mathbf{u} \leq \varepsilon \tag{18}$$

and at the same time the Riemannian approximating sums of this integral do not exceed ε if the diagonal of the subdivision is small enough. Therefore, as $\frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|} \leq 1$,

$$\frac{1}{\Lambda_n^d} \sum_{\mathbf{u} \in 2\mathcal{G}_n^{\mathbf{0}} \setminus M_\varepsilon} \frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|} \|g_{\mathbf{u}}\| \leq \varepsilon \,, \tag{19}$$

when $\frac{1}{\Lambda_n^d}$ is small enough, i.e. when n is large enough: $n \geq n_\varepsilon$. Fix ε, M_ε and assume that $n \geq n_\varepsilon$. Because $g_{\mathbf{u}}$ is Riemann integrable as a function $g : \mathbb{R}_{\mathbf{0}}^d \to C(\mathbb{R}^2)$ on R for each bounded closed d-dimensional rectangle R in $\mathbb{R}_{\mathbf{0}}^d$, therefore we have

$$\Big\| \frac{1}{\Lambda_n^d} \sum_{\mathbf{u} \in 2\mathcal{G}_n^{\mathbf{0}} \cap M_\varepsilon} g_{\mathbf{u}} - \int_{M_\varepsilon} g_{\mathbf{u}} \, d\mathbf{u} \Big\| \leq \varepsilon \tag{20}$$

in the space $C(\mathbb{R}^2)$, if n is large enough. This relation and (18) imply that $\int_{\mathbb{R}_{\mathbf{0}}^d} g_{\mathbf{u}}(x,y) \, d\mathbf{u}$ exists and it is continuous in (x,y). As each edge of G_n converges to ∞, $\frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|} \to 1$ uniformly according to $\mathbf{u} \in M_\varepsilon$. Therefore, using that $\|g_{\mathbf{u}}\|$ is directly Riemann integrable, we obtain that

$$\Big\| \frac{1}{\Lambda_n^d} \sum_{\mathbf{u} \in 2\mathcal{G}_n^{\mathbf{0}} \cap M_\varepsilon} \frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|} g_{\mathbf{u}} - \frac{1}{\Lambda_n^d} \sum_{\mathbf{u} \in 2\mathcal{G}_n^{\mathbf{0}} \cap M_\varepsilon} g_{\mathbf{u}} \Big\| \leq \varepsilon \,, \tag{21}$$

if n is large enough.

Relations (18), (19), (20), and (21) imply that

$$\Big\| \frac{1}{\Lambda_n^d} \sum_{\mathbf{u} \in 2\mathcal{G}_n^{\mathbf{0}}} \frac{|\mathcal{G}_{n,\mathbf{u}}|}{|\mathcal{G}_n|} g_{\mathbf{u}} - \int_{\mathbb{R}_{\mathbf{0}}^d} g_{\mathbf{u}} \, d\mathbf{u} \Big\| \leq 4\varepsilon \,, \tag{22}$$

if n is large enough.

Therefore, using that $\frac{1}{h_n} K\big(\frac{x_r - u}{h_n}\big)$ is a density function, we have

$$\left| B - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Big\{ \frac{1}{h_n} K\Big(\frac{x_r - u}{h_n}\Big) \frac{1}{h_n} K\Big(\frac{x_s - v}{h_n}\Big) \int_{\mathbb{R}_{\mathbf{0}}^d} g_{\mathbf{u}}(u,v) \, d\mathbf{u} \Big\} \, du dv \right| \leq 4\varepsilon \,, \tag{23}$$

if n is large enough. As $\int_{\mathbb{R}_{\mathbf{0}}^d} g_{\mathbf{u}}(u,v) \, d\mathbf{u}$ is continuous according to (u,v), the limit of the double integral in this expression is $\int_{\mathbb{R}_{\mathbf{0}}^d} g_{\mathbf{u}}(x_r, x_s) \, d\mathbf{u} = \sigma(x_r, x_s)$.

Finally, we have to prove that in (9) $\mathbb{E} f_n(x_i)$ can be changed for $f(x_i)$. To this end we have to prove that $\sqrt{|\mathcal{D}_n| / \Lambda_n^d} \big(\mathbb{E} f_n(x) - f(x)\big) \to 0$. This is valid because, by (10), $h_n^{-2} \big(\mathbb{E} f_n(x) - f(x)\big)$ is bounded, and $\sqrt{|\mathcal{D}_n| / \Lambda_n^d} h_n^2 = \sqrt{|T_n|} h_n^2 \to 0$. $\square$

4. Examples

In this section we present a simple example that gives numerical evidence for the phenomenon described in Theorem 2.1.

Let $\xi_{\mathbf{u}}$, $\mathbf{u} \in \mathbb{R}^d$, be a stationary Gaussian random field with mean value function zero and with covariance function $r_{\mathbf{u}}$. Assume that $r_{\mathbf{u}}$ is continuous and $r_{\mathbf{0}} = 1$. Therefore $r_{\mathbf{u}} \to 1$, as $\|\mathbf{u}\| \to 0$. It is plausible to assume that $r_{\mathbf{u}} \to 0$, as $\|\mathbf{u}\| \to \infty$. We need the joint density function of $\xi_{\mathbf{0}}$ and $\xi_{\mathbf{u}}$, $\mathbf{u} \neq \mathbf{0}$. Therefore we have to assume that $|r_{\mathbf{u}}| < 1$ if $\mathbf{u} \neq \mathbf{0}$. We strenghten this and assume that outside a neighbourhood of the origin $|r_{\mathbf{u}}| \leq c < 1$. Assume also that $r_{\mathbf{u}} \neq 0$, $\mathbf{u} \in \mathbb{R}^d_{\mathbf{0}}$.

For any particular field $\xi_{\mathbf{u}}$, we have to check if the conditions of Theorem 2.1 are satisfied. Here

$$g_{\mathbf{u}}(x, y) = f_{\mathbf{u}}(x, y) - f(x) f(y)\,, \tag{1}$$

$$f_{\mathbf{u}}(x, y) = \frac{1}{2\pi\sqrt{1 - r_{\mathbf{u}}^2}} e^{-\frac{1}{2}(ax^2 + ay^2 - 2bxy)}\,, \quad f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\,,$$

where $a = 1/(1 - r_{\mathbf{u}}^2)$, $b = r_{\mathbf{u}}/(1 - r_{\mathbf{u}}^2)$. Now we study if the improper Riemann integral $\int_{\mathbb{R}^d_{\mathbf{0}}} \|g_{\mathbf{u}}\|\, d\mathbf{u}$ exists and is finite. This is satisfied for the covariance function $r_{\mathbf{u}}$, if

$$\int_O \frac{1}{\sqrt{1 - r_{\mathbf{u}}^2}}\, d\mathbf{u} < \infty \tag{2}$$

for a (closed, bounded) domain O containing a neigbourhood of the origin; and

$$\int_N \frac{1}{\sqrt{1 - r_{\mathbf{u}}^2}} - 1\, d\mathbf{u} < \infty, \qquad \int_N \left| \frac{-r_{\mathbf{u}}^2 \pm r_{\mathbf{u}}}{1 - r_{\mathbf{u}}^2} \right| d\mathbf{u} < \infty \tag{3}$$

for N being the complementer of a bounded neigbourhood of the origin.

Example. Consider a Gaussian field $\xi_{(u,v)}$, $(u, v) \in \mathbb{R}^2$, with mean value function zero and covariance function $r_{(u,v)} = e^{-(|u|+|v|)}$, $(u, v) \in \mathbb{R}^2$. This function satisfies the above mentioned conditions (2)–(3).

Example. (Simulation results.) Consider the Gaussian process $\xi(u)$, $u \in \mathbb{R}$, with mean zero and covariance function $r_u = e^{-|u|}$, $u \in \mathbb{R}$. This function satisfies conditions (2)–(3). The direct Riemann integrability of $\|g_{\mathbf{u}}\|$ is also satisfied.

We observe this process in the $1/\Lambda$-lattice points of the domain $T = [0, t]$ with $\Lambda = 200$ and $t = 100$. That is the sample is $z_1 = \xi(1/200), \ldots, z_s = \xi(20000/200)$ with $s = 20000$. Now the covariance matrix of this data vector is $(r^{|i-j|})_{i,j=1}^s$, where $r = e^{-1/\Lambda}$. Therefore the data generation for the simulation is easy. Let $y_1, \ldots, y_s$ be i.i.d. standard normal and choose $z_i = r^{i-1} y_1 + \sqrt{1 - r^2} \sum_{j=2}^{i} r^{i-j} y_j$, $i = 1, \ldots, s$.

Using this data, we gave kernel estimation for the density function of the process (i.e. the standard normal density function). We calculated the estimator at points $x_1 = -2$, $x_2 = -1$, $x_3 = 0$, $x_4 = 1$, $x_5 = 2$. We used values of the bandwidth: $h_1 = 0.01$ and $h_2 = 0.001$. We applied the standard normal density function as kernel K.

The simulations were performed with MATLAB. 500 repetitions were made. The data sets for $h_1 = 0.01$ and $h_2 = 0.001$ were the same. The theoretical values of the density function and the averages of their estimators are shown in the following table.

	x	-2	-1	0	1	2
	$f(x)$	0.0540	0.2420	0.3989	0.2420	0.0540
$h = 0.01$	estimators' mean	0.0543	0.2418	0.4003	0.2414	0.0518
$h = 0.001$	estimators' mean	0.0528	0.2422	0.4026	0.2445	0.0492

Then we calculated the empirical covariance matrices of our standardized estimators (according to equation (9), the standardization factor is $\sqrt{\frac{|\mathcal{D}|}{\Lambda}} = 10$).

$$\Sigma_1 = \begin{bmatrix} +0.0683 & +0.0320 & -0.0308 & -0.0444 & -0.0135 \\ +0.0320 & +0.2307 & -0.0258 & -0.1356 & -0.0453 \\ -0.0308 & -0.0258 & +0.2442 & -0.0355 & -0.0284 \\ -0.0444 & -0.1356 & -0.0355 & +0.2421 & +0.0388 \\ -0.0135 & -0.0453 & -0.0284 & +0.0388 & +0.0598 \end{bmatrix};$$

$$\Sigma_2 = \begin{bmatrix} +0.1339 & +0.0379 & -0.0296 & -0.0451 & -0.0162 \\ +0.0379 & +0.5481 & -0.0085 & -0.1544 & -0.0482 \\ -0.0296 & -0.0085 & +0.7092 & -0.0416 & -0.0205 \\ -0.0451 & -0.1544 & -0.0416 & +0.6002 & +0.0459 \\ -0.0162 & -0.0482 & -0.0205 & +0.0459 & +0.1116 \end{bmatrix}.$$

Covariance Σ_1 corresponds to bandwidth h_1 while Σ_2 corresponds to bandwidth h_2. The difference of the diagonals of Σ_2 and Σ_1 seems to be significant.

Now calculate the additional terms of the covariance matrices described in Theorem 2.1. In our case

$$\frac{1}{\Lambda}\frac{1}{h} f(x_i) \int_{-\infty}^{+\infty} K^2(u)\, du = \frac{1}{200}\frac{1}{h} f(x_i)\, \frac{1}{2\sqrt{\pi}}.$$

Therefore the diagonal elements of the matrix D in Theorem 2.1 for $h_1 = 0.01$ and $h_2 = 0.001$ are the following:

$$\begin{aligned} \mathrm{diag}D_1 &= \begin{bmatrix} 0.0076 & 0.0341 & 0.0563 & 0.0341 & 0.0076 \end{bmatrix}; \\ \mathrm{diag}D_2 &= \begin{bmatrix} 0.0762 & 0.3413 & 0.5627 & 0.3413 & 0.0762 \end{bmatrix}. \end{aligned}$$

As in the infill-increasing case only the diagonals of the limit covariance matrices can be different for different values of the bandwidth, we show in the following table the diagonal of the differences of the empirical and that of the theoretical covariance matrices.

$\mathrm{diag}(D_2 - D_1)$	0.0686	0.3072	0.5064	0.3072	0.0686
$\mathrm{diag}(\Sigma_2 - \Sigma_1)$	0.0656	0.3174	0.4649	0.3582	0.0517

The results show that the diagonal matrix D of Theorem 2.1 explains well the dependence of the limit covariance matrix on the bandwidth.

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