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"Synthetic Mathematics" Maths & Geometry: A Symbiosis with Misunderstandings

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Abstract

The article is a plea for keeping classical Geometry in syllabuses of mathematics and engineering studies. Arguments are given for seeing Geometry as an independent art of human culture, rooting in Philosophy and Art more than in the needs of craft and surveying. Some examples of inadequate mathematical concepts and violations of visualisation logic aim at showing that insufficient education in (mathematical and geometric) fundamentals impede further development of Mathematics and Informatics.

Categories and Subject Descriptors: geometry, education, history of geometry

Key Words and Phrases: geometry in history and art, visual logic, synthetic reasoning, geometry education

1. Introduction

In this article a very personal point of view about the relationship between Mathematics and Geometry is given. There seems to be indications for misunderstanding Geometry simply as a part of Mathematics and neglecting its independent role for human culture, for a misinterpretation of its historical roots, for a low regard for its "visual logic" and for neglecting the consequences of an insufficient education of engineering science students in Geometry and Graphics. This paper wants to add perhaps new facets to the ongoing discussion about the future of Geometry education. Especially now, in times of economical and political unification in Europe and the tendencies to standardise school and university syllabuses, all geometers of different countries should work together in their efforts to convince decision-makers of the economical and cultural advantage of having a competent and broad education in Geometry and Graphics. In the following some arguments for Geometry as a subjects for its own somehow equidistant to Mathematics and Informatics are given by means of selected topics as:

- redefining Geometry from its historical context
- "synthetic argumentation" as a general scientific method
- interaction between Geometry and Arts
- misleading mathematical concepts and figures
- restricted creativity by CAD-monopolists
- consequences of cancelling geometry in education.

2. Redefining Geometry from its historical context

It is widely common opinion that Geometry arose from practical needs in early cultures, as for example the annual flooding by the Nile river made it necessary to re-survey meadows and fields. When translating the Greek components of the word "geometry" as "measure (of) land", such a point of view is indeed indicated.

The 2002 flooding of Dresden, Germany, and similar catastrophes at other places in Europe together with a experiences from a visit in Egypt convinced me that it is not at all necessary to re-survey land covered with a layer of mud of maximum 1cm thickness. By the way, most of the farm land in ancient Egypt belonged to temple communities, and taxes were imposed according to the crop and not according to the size of the land. There is no need for extraordinary accuracy in surveying at that time. And even now, in agriculture, the border strips of meadows and fields are rather extensively used as paths to other fields. It simply does not depend on one Inch or two.

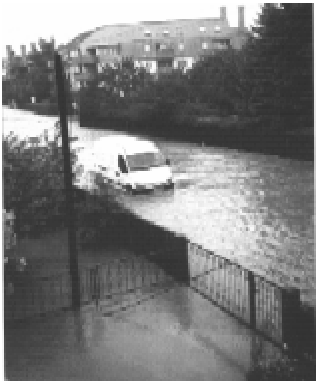


Figure 1: Floodings in Europe leaving some mm of mud

On the contrary, Egyptian temples, like that in Carnac/Luksor surely were not built as ad-hoc constructions, but needed exact planning. Forming such big stones to a wall brick puzzle without gaps needs methods of exact measurements not only of distances, but also of (right) angles. Becoming better acquainted with Egyptian religion and Greek philosophy one learns that exactness of edges and faces of such stones and walls as well as symmetry and special measure ratios of temples represent the divine principle. Incompleteness and disorder represents the earthly side of life. Obelisks are not only fascinating, when erected pointing to the transcendent world in heaven, but their absolute plane facets and unbelievable straight edges additionally symbolise the transcendent world, too.

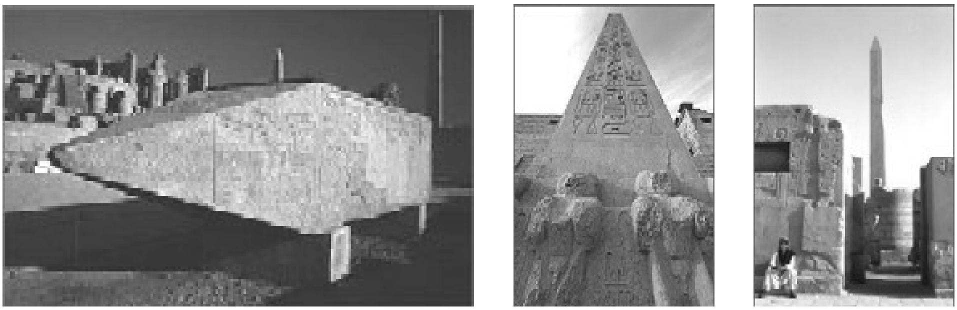


Figure 2: Egyptian temples (Carnac): monuments of geometric ideas and accuracy

Considering these facts and impressions one will find the following etymology of “Geometry” more likely than the common one:

Geometry is a composition of

$\gamma\alpha\iota\alpha$, meaning the goddess of the whole earth/world, and of

$\mu\varepsilon\tau\rho\nu$, primarily meaning “metre” (for poems) and (inner) symmetry.

Thus *geometry* means “*view of the world*” in a religious/philosophical sense rather than surveying.

This view also gives sense to the door plate of Platon's Academy of Athens, which roughly translated warned:

No Entrance for Geometry-Ignorants!

If Platon and his contemporaries had understood Geometry simply as a useful and applicable craft, he surely had not used Geometry as an entrance restriction to his philosophical school.

By the way, the door plate mentions "Geometry", not "Mathematics"!

Figure 3: Platon's "Academy of Athens" as painted by Raphael

3. Science "modo geometrico"

It is well-known that Euclid's famous "Elements", by its axiomatic structure, influenced the development of natural and human sciences from the 16^{th} century onwards. E.g. Baruch Spinoza (1632–1677), and Thomas Hobbes (1588–1679) are outstanding representatives of the 'geometric method', namely deducing ethical and philosophical and even juridical statements from a system of basic "axioms" via logical reasoning. This method of reasoning, based on logic rules applied to fundamental objects and theorems, is "synthetic" in the truest sense of the word, as new objects and theorems are synthesised by sequels of old ones.

Contrasting to that, nowadays mathematical semiliterates often combine the word "synthetic" with "geometry" and mean the somehow nice but hopelessly old-fashioned "geometry of triangles". Sometimes they want to defend this Synthetic Geometry against the more effective "analytic method"[1] and propose a revival of Elementary Geometry.

Without going into further details of sciences, which have been treated modo geometrico and without stressing the differences between both, the analytic and the synthetic method in Mathematics, I believe to be right when stating:

[1]An attempt in that direction was a lecture at "Sächsischer Geometrietag 2003", (Magdeburg, Germany), by a physicist.

“Synthetic reasoning” is a general method of argumentation, it is rather synonymous to “logical thinking” than a method restricted to pre-Cartesian (co-ordinates free) Geometry.

One of the crucial topics for the development of modern Physics and Mathematics is the Euclidean Parallel Postulate. The struggle for finding it independent of the other axioms or not was carried out by synthetic reasoning and finally lead to the discovery of non-euclidean geometries. This influenced and changed the view of the world not only of a few scientists, but finally of everybody: Einstein, Lorentz and Minkowski showed that many aspects of our real world can (at least locally) be *modelled*[2] by a special type of a non-euclidean geometry. Unnecessary to say that this changed and stimulated technologies in the world around us irreversibly. Further details on the history of Geometry see [12].

Again one may state that at the beginning of all modern technology stands Geometry. And Geometry, via understandable models, accompanies that development up to now.

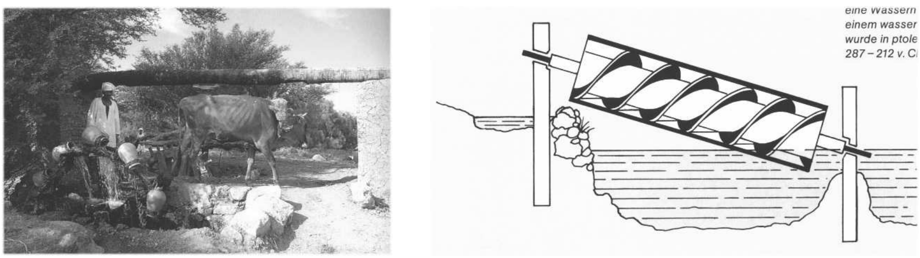


Figure 4: Early technologies: a symbiosis of Geometry and Mechanics

4. Interaction between Geometry and Arts

Besides of logic and abstraction Geometry was and is nourished not only by practical needs of Architecture and Engineering. Also (decorative) Art is a rich source for Geometry. Here comes that both, Art and Geometry have 2D- or 3D-visualisation in common as a more or less unavoidable medium. One can say that visualisation is characteristic for Fine Arts and an essential part of Geometry. Also here history shows that visualisation methods were developed in personal union of artists and geometers (who were not mathematicians) like Piero della Francesca (1415?–1492) and Albrecht Dürer (1471–1528).

In fact, constructive methods dating back to these famous painters allow us today to “control” the accuracy of e.g. the “Melencholia II” of Dürer without using any calculation. Such an analysis by descriptive geometric methods is given by e.g.

[2]One could be seduced to replace the word “modelled” by “understood”. In the following it will come clear that these concepts are not at all identical! We understand facts only so far we understand a model.

E. Schröder [11] and H. Schaal [10], and recently by G. Baer [2].[3]

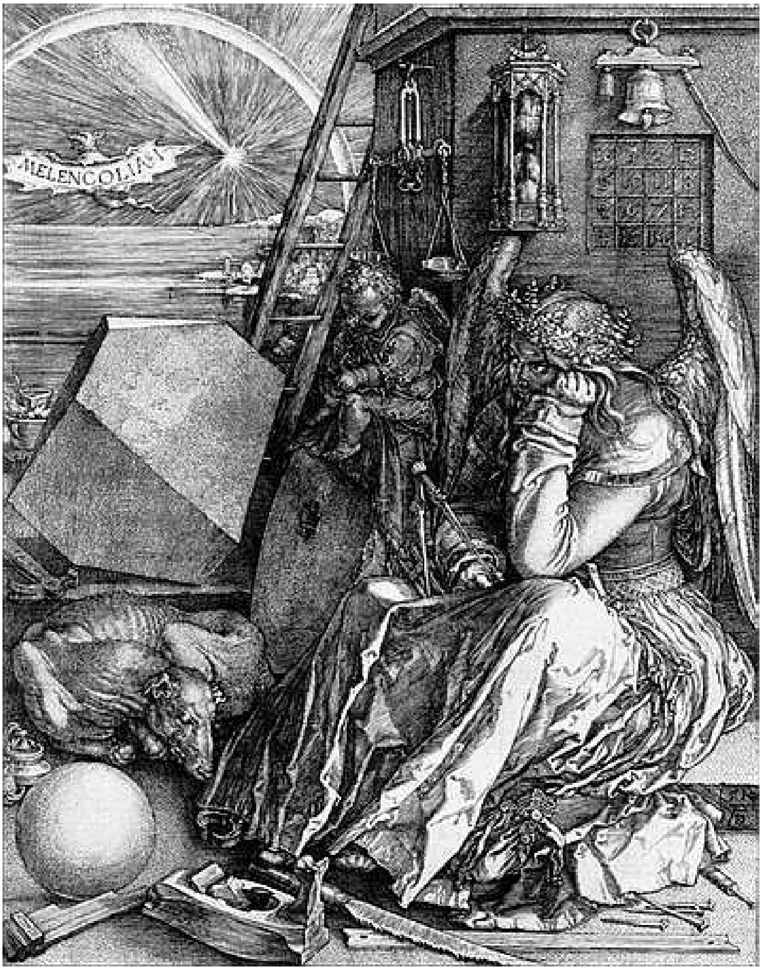


Figure 5: Dürer's "Melancholia II": manifestation of Geometry as a view of world?

When artists depict or sculpt geometric objects, e.g. polyhedra like the one in Dürer's "Melencholia II", it is not only because they are aesthetically appealing, but also because they realise abstract laws and inner symmetries. They carry an abstract, often non-mathematical message, which unfortunately remains hidden for those not in the know. The Platonic solids are classical examples of such

[3]It is not surprising that the "inventors of geometric perspective drawings" used their know-how as far as possible. So rather more interesting is the question, where Dürer consciously ignored the rules of a perspective and made concessions to the viewer due to conventions or better understanding. E.g. in his "Melencholia II" the ball, a classical symbol for completeness, is shown with a circular circumference instead of the "right" elliptic one.

objects: They where symbols for the ancient Greek “atoms”, air, fire, water, earth and heaven; in his ’Mysterium Cosmographicum’ Johannes Kepler (1571–1630) combines these solids with the orbits of the then known planets. Modern versions like the cube (= holy Kaaba(!)) also make use of psychological effects, see Figure 6.

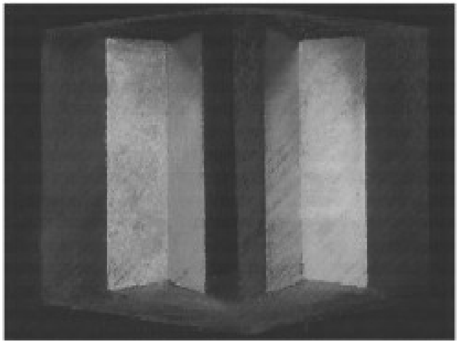


Figure 6: Geometric stage setting: the cube as the cage of life for Tristan and Isolde

Ornaments are used in all human cultures and their main purpose is to enjoy the eye of the viewer. But on closer inspection the most pleasure it seems to be the discovery of symmetries of a sophisticated ornament, which gives.

As an interaction the classification and realisation of all types of wallpaper groups arose as a mathematical problem with the urge for generalisations to three and higher dimensions. Also here Geometry occurs as mediator between Art and Mathematics.

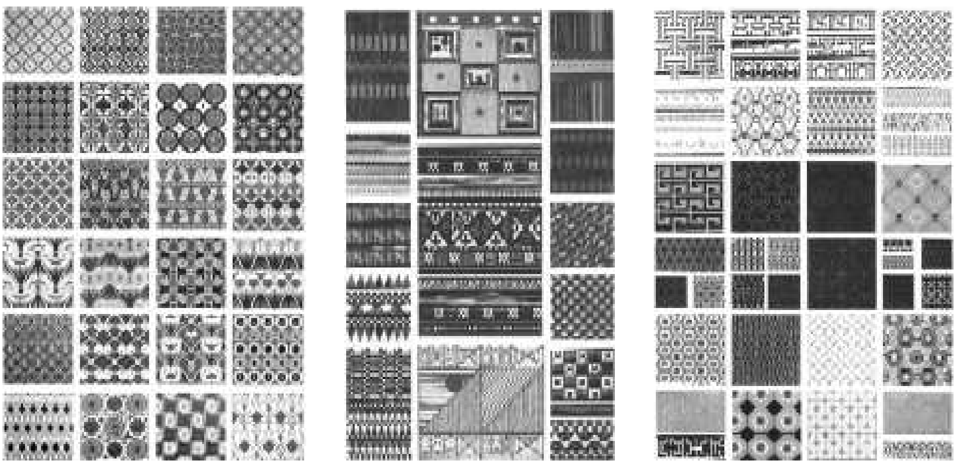


Figure 7: Wallpaper ornaments in Ancient Egypt, Oceania and China

Conversely the popularisation of the mathematical concepts of four dimensions also influenced artists to visualise objects of such a space or symbolise the (3,1)-space-time itself.

An artist, who dedicated himself intensively to that topic, was Salvadore Dali (1904-1989): In his “Crucifixus hypercubicus” (Figure 8) he depicts an exact perspective of the 'net' of a hypercube of the 4D-space. Many other paintings show 3D-scenes with melting watches symbolising the time-dimension.

Polyhedra as spacefilling objects or as shaky or movable structures have been investigated by many authors. I just restrict the references here to [9], [13], [5].

Summing up this chapter one may conclude:

The nearness of Geometry to (decorative) Arts induces that geometric visualisation also has non-mathematical (esthetical and psychological) components, and this is characteristic for Geometry. Objects of Fine Art often stimulate geometers to further investigation. The starting point for such research is not seldom a good guess won just by looking at figures and models.

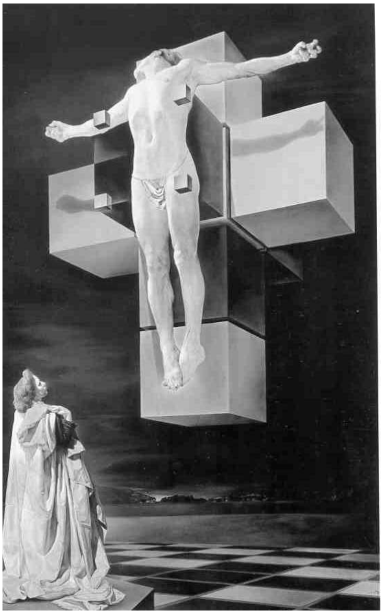


Figure 8: “Crucifixus hypercubicus” (S. Dali): perspective drawing of the net of a hypercube in 4-space

5. Misleading mathematical concepts and “wrong” figures

Modern mathematical disciplines and Geometry have developed a specific terminology. Sometimes concepts, coined in history, now have different interpretations.

Some concepts are misleading and even conceptually wrong (due to history, to translation or simply to ignorance of use in other connotations). To facilitate the intercourse of scientists one should think of introducing more suitable concepts. Some few examples:

In most books of American or English origin dealing with complex numbers one will find the concept "complex plane" meaning the classical $\mathbb{R}^2$-model represented by a euclidean (and thus real) plane endowed with a cartesian frame, see e.g. [1] and [4]. It is conceptually wrong to call the one-dimensional set of complex numbers $\mathbb{C}$ a (complex) plane, as long as Algebraic Geometry uses that term correctly when investigating algebraic curves in a projectively extended complex plane. The Riemannian sphere S^2 is indeed a "complex (algebraic) manifold", but this interpretation of the term is not meant in [Fra], where S^2 is seen as a model for $\mathbb{C} \cup \infty$. The complex points of S^2 are not images of complex numbers or points of $\mathbb{R}^2$.

Quite another interpretation of the term "complex (projective) plane" can be found in [3], where that plane automatically is endowed with a unitary inner product and therefore should be called "elliptic complex plane".

Mathematicians use the term "projection" mostly in the sense of "normal projection", thus assuming a (euclidean) inner product space, while Geometers understand "projection" as linear operation in general projective spaces.

Some misunderstandings result from mixing up concepts of different models for one structure: Topological objects, like "sphere", "cylinder" or "torus", occur together with differential geometric objects with the same names. Sometimes, e.g. in the Geometry of Minkowski planes[4], one pretends to talk about the naked point set of say the plane of intuitive view, but uses the canonical inner product in $\mathbb{R}^2$ and co-ordinates to define the dual Minkowski plane. If furthermore original and dual "unit" are identified, one gains interesting theorems in Minkowski Geometry. But these theorems have meaning only in a Minkowski Geometry endowed with a euclidean base structure and not in a pure Minkowski Geometry.

Sometimes identifications of an analytic model with a point model (as e.g. the plane of intuitive view) show more than wanted, sometimes less. In the analytic model 0 and 1 is canonically given, in the point set a "unit" and a co-ordinate frame have to be arbitrarily defined. Such a scene is no longer a plane or space of intuitive view.

Visualising mathematical objects and structures is, to my opinion, a sort of modelling and should be logically correct at least in a topological sense. When looking in common text books on Geometry or mathematics formularies (see e.g. [1], [4]), written by mathematicians, it is striking, how many attempts to draw simple figures go wrong! It seems that *visualisation logic* differs from *mathematical logic*. If so, Geometry cannot simply be a part of Mathematics.

Summarising the chapter we can conclude:

There is an increasing lack of accuracy in dealing with mathematical/geometric concepts and models. Not distinguishing the languages and the ranges of differ-

[4]Minkowski spaces are real affine spaces with a (symmetric) distance function based on an arbitrarily given, centrally symmetric convex body as the unit ball.

ent models for certain abstract facts may cause confusion and make learning and understanding unnecessarily difficult for students. Mathematical illustrations and figures often neglect logic rules; visualisation logic should not be put on the lowest level of logic-applications.

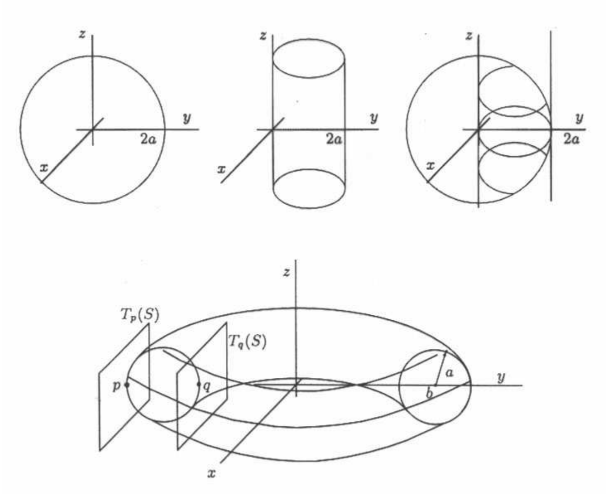


Figure 9a: Visualisation of the Viviani-intersection of sphere and cylinder (above) and of a torus

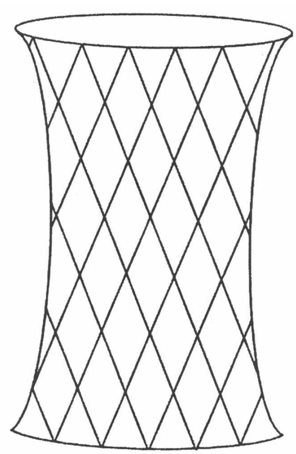
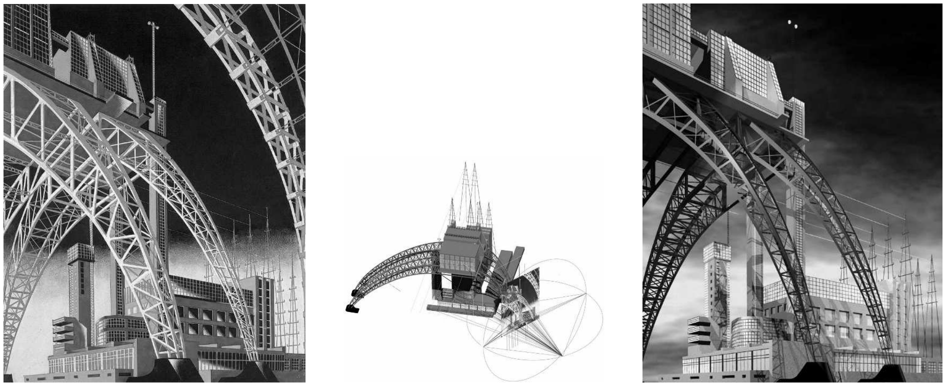


Figure 9b: Visualisation of a onesheeted hyerboliod

6. Restricted creativity by CAD-monopolists

As a co-adviser during practical training of mathematics students in connection with software producing firms I notice that industry sometimes prefer approximative solutions rather than exact ones based on geometric considerations. In one example, the computer aided automatic 3D-reconstruction from perspective sketches drawn by hand ignored the fact that such a sketch is only affine and not similar to an ordinary perspective drawing. To receive similarity a certain criterion (see [16]) has to be fulfilled. Based on a geometric approach one could receive high quality reconstructions by reasonable expenditure, see e.g. [8]:



(a)

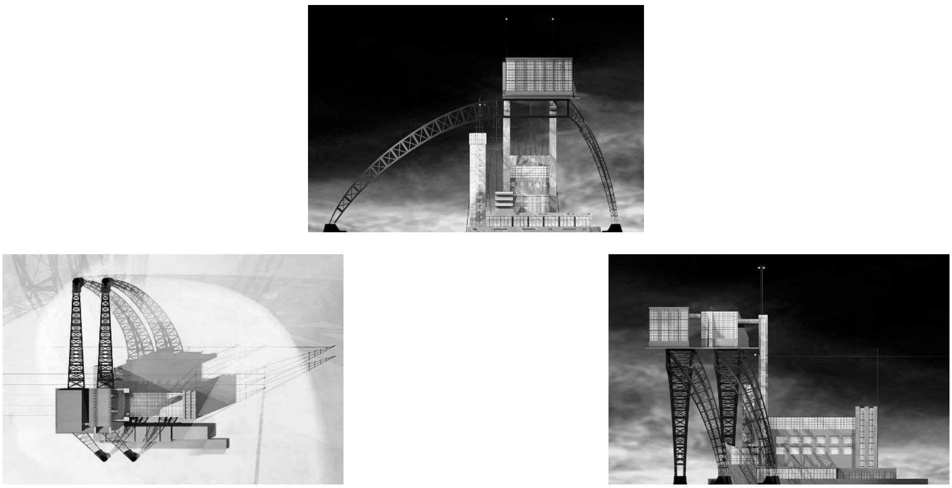


Figure10: Tschernichow’s ideal architecture: hand painting original (a) and 3D-reconstructions

Computer Vision and automatic optical quality control have to “invent” methods of (linear) mapping for their own. This highly applicable topic is up to now not fully covered in modern textbooks on Geometry, although high-school teachers in Descriptive Geometry are educated in that field. But even if such a textbook or course would exist, to benefit from it the software developer had to be acquainted with Projective Geometry. Time restricted syllabuses for studies in Informatics and all sort of Mathematics have to do the splits between introducing the student to actual research and at the same time cultivating a solid basis of fundamentals.

For applicants there seems to be a need of articles like [15] as a consequence that

our syllabuses now are neglecting a deeper and precise understanding of fundamentals in Mathematics and Geometry and in Informatics as well.

7. Consequences of cancelling geometry in education

If the last statement above were right to a larger extent, it would cause that University lecturers by themselves, at least in the near future, lack precise understanding of fundamentals. This would finally lead to unefficient and unimportant research, too.

It is somehow strange, that Didactics in Mathematics and Informatics only aim at education in school and that there are, at least in my environment and to my own experience, no attempts to deal methodical questions for maths courses at university level. People teach what they know and often the way they were taught. It is somehow natural to believe that, what we do not know, cannot be so important, otherwise we would have learned it during our own studies. As nowadays high-schools in general do not provide university freshmen with solid geometric and mathematical fundamentals anymore, we should direct extraordinary attention to a solid basis education. And here belongs an education in Geometry, too.

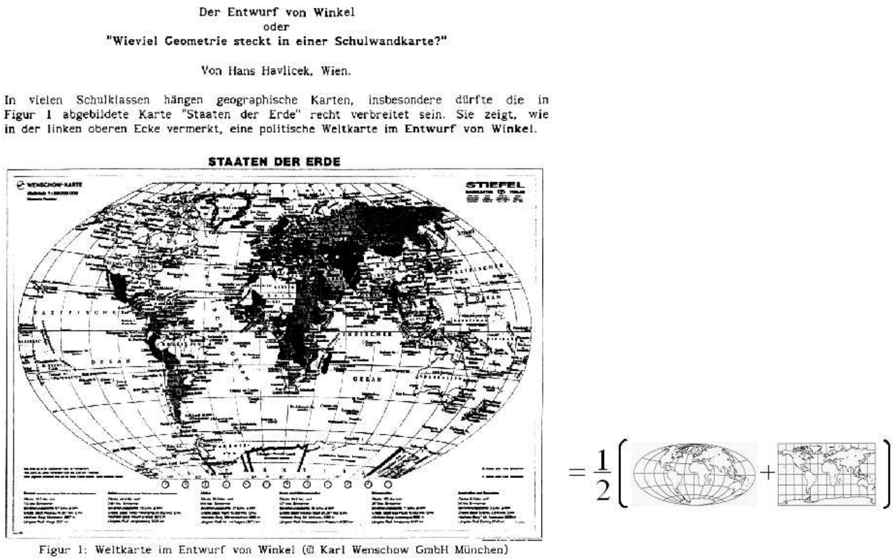
The ongoing impoverishment of basic and higher knowledge in Geometry since the 1960ies resulted from N. Bourbaki's view that any geometric problem can be transformed to a mathematical one. Although this point of view was and is very fruitful for some mathematical disciplines, e.g. for Fundamentals of Geometry, it neglects the independent role of (classical) geometry for visualisation and for providing Mathematics and Science with proper models.

Geometry helps to systemise and generalise mathematical problems. Such an important and fruitful systemisation is e.g. the Erlangen Program (1872) of F. Klein.

Another example is again the Theory of (linear) Mappings: F. Hohenberg discovered a method of merging two parallel projections of an object to a third one without auxiliary lines, see [7]. This principle can be used in a broad spectre of applications, see e.g. [6] and [18].

Now maths syllabuses cancel such basic geometric skills as e.g. Projective (and Affine) Geometry, and $\mathbb{R}^n$ solely replace affine and projective spaces A^n or $P^{(n-1)}$. But this $\mathbb{R}^n$ is not the space of the engineer or architect, where "points" are points and not vectors.

Because of their educational background mathematicians have no interest/competence in graphics education of engineers and architects.



Der Entwurf von Winkel
oder
"Wieviel Geometrie steckt in einer Schulwandkarte?"

Von Hans Havlicek, Wien.

In vielen Schulklassen hängen geographische Karten, insbesondere dürfte die in Figur 1 abgebildete Karte "Staaten der Erde" recht verbreitet sein. Sie zeigt, wie in der linken oberen Ecke vermerkt, eine politische Weltkarte im Entwurf von Winkel.

Figur 1: Weltkarte im Entwurf von Winkel (© Karl Wenschow GmbH München)

Figure 11: The Hohenberg principle of merging two (linear) images to a third one, modified for mergingearth maps

Engineers are weakly educated in Geometry/Graphics, they get no deeper insight in the scientific part of these topics.

If they have to teach Geometry and Graphics, they mostly stick to what they have learned as a student, what finally leads to a stagnation in the methodology of Geometry and Graphics.

Obviously Computer Aided Design became an unavoidable tool for teaching Geometry. But it is a tool, not Geometry itself and it needs a specialised methodical approach to draw advantage of that tool in developing spatial abilities and in the creative handling of geometric objects. Without educated teachers (at school and university level) "Geometry" becomes trivialised and reduced to what commercial CAD-systems and Automatized Problem Proving provide via handbooks.

Summarizing we can state that:

Reductions of Geometry and Graphics education at university level has consequences not only for engineering and architecture studies, where basic geometric skills are unavoidable. Also Mathematics and Informatics have geometers as import partners for "translating" say an explicite industrial problem to a mathematically solvable one. The consequences of widely cancelling Geometry in Mathematics syllabuses and cancelling further education of teachers in Geometry will be seen not at once, but when most of the relatively good educated teachers will retire. Then future teaching of Geometry will become rather a matter of compensation than a matter of What and How.

8. Conclusion

Collecting the preceding statements and extracting the essence we find that

- Geometry is an expression of human culture for its own.
- Geometry applies Mathematics and Informatics in the same manner as Theoretical Physics applies Mathematics.
- Geometry stimulated development in Mathematics.
- “Synthetic” in connection with “Geometry” should not evoke the co-notation ’old fashioned Elementary Geometry’. Contrasting to that ’synthetic’ means a general method of deducing statements via (more or less naive) logic.
- Geometry provides Mathematics with intuitively understandable models and helps to ’control’ mathematical concepts. Some Mathematical concepts are not at all ’logical’ and should be improved.
- Geometry provides Mathematicians and Engineers with basic skills in visualisation problems. Visualisation logic is widely violated by mathematicians with less or no education in classical Geometry.
- Geometry is an ideal link between a technical problem and its mathematical solution.
- Reduction of Geometry in syllabuses has negative influence in the education of engineers and architects and consequently it has negative influence to a part of human culture and environment.
- Development in Geometry (including computer aided applications) is more likely if based on broader education.
- The key for such a broader education in classical Geometry is the education and further education of teachers on both school and university level.
- Concluding It is worthwhile to join an international lobby aiming at preserving at least a minimum of classical geometry in maths (and engineering) syllabuses.

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