

Construction ruled surfaces from scattered data by Kohonen network

Emőd Kovács

Department of Information Technology, Eszterházy Károly College
e-mail: emod@ektf.hu

Abstract

In this paper a new approach is presented, where first an artificial neural network is used to order the data and form a grid of control vertices with quadrilateral topology, hence after this step the original, well-known free-form methods can be applied to construct the ruled surface. The Plücker coordinates of the projective lines will be also applied.

Key Words and Phrases: Ruled surface, NURBS, Kohonen neural network

1. Introduction

Ruled surfaces and their special subclass, developable surfaces are well-known in classical geometry, but also have extended applications in computer geometry and graphics as well as in computer aided manufacture. In these latter fields, however, the standard way of describing surfaces is the different types of spline-surfaces, like Bézier, B-spline or NURB surface. Hence it is highly desired to describe and construct ruled and developable surfaces as spline surface. Several methods have been developed to approximate or interpolate such kind of surfaces, see e.g. [2], [10]. In our paper original input data structure consists of a set of given lines, called rulings. In this case the description of these lines and the neural network by homogeneous coordinates will be advantageous, which is a well-known technique in projective geometry [1]. The Plücker coordinates of the projective lines will be also applied. The purpose of this paper is to improve and extend the method of modeling scattered data by free-form surfaces. In that method an artificial neural network, the Kohonen network was used for order the data and form a quadrilateral control grid from the scattered points, hence the standard free-form methods, like Bézier surface or NURBS, could be applied to approximate or interpolate the data.

The standard definition of any type of spline surfaces uses a quadrilateral control grid as input data, which is a $(2, n)$ type in case of ruled surfaces. The main purpose of most of the known methods is to construct this control grid. In our method

the Kohonen neural network has been applied in a preprocessing step to produce a control grid from the original input data. This artificial intelligence tool has already been applied successfully by the authors in other approximation problems [6],[7].

In our paper original input data structure consists of a set of given lines, called rulings. In this case the description of these lines and the neural network by homogeneous coordinates will be advantageous, which is a well-known technique in projective geometry. In another sense this computer geometrical approach has already been developed by Ravani et al. in [1]. The Plücker coordinates of the projective lines [5] will be also applied, which is investigated by Chen and Pottmann [2].

2. Construction ruled B-spline surfaces by Kohonen neural network

We begin our discussion with the definition of ruled surfaces.

Definition 2.1. A surface is ruled surface, if through every point of the surface there passes a line which lies entirely on the surface. These lines are called rulings of the surface.

This means, that a ruled surface is covered by a one parameter set of lines, the rulings. These lines have the following property: a ruling lies in the tangent plane of the surface at every point of the ruling. This property is important in terms of the special ruled surfaces, called developable surfaces.

Theorem 2.1. *A ruled surface is developable, if and only if the tangent planes at all points of an arbitrary ruling coincide.*

If these tangent planes vary along the ruling, the surface is called general (non developable) surface. In this paper we will consider only general ruled surfaces.

Consider the following problem: an arbitrary set of lines are given, find a ruled surface passing these lines, that is a surface for which the given lines are rulings.

More precisely, we would like to interpolate these given lines with one of the standard spline surfaces of CAD, a Bézier, a B-spline or a NURB surface. In this paper the B-spline surface will be used, but all of these surfaces are defined by a control grid of points, called control points. This grid regularly has a quadrilateral topology, and one can chose to approximate or to interpolate these points.

Theorem 2.2. *If the points of the control grid of a B-spline surface form straight lines in one direction and the type of the grid is $(2, n)$, then the result surface is ruled surface.*

Thus we need to form a quadrilateral grid in which the vertices form straight lines in the direction of the given lines. These lines will be the rulings of the future surface, if it will interpolate the control points, and the theorem mentioned above

yields it will be a ruled surface. To find this grid the Kohonen neural network will be applied. This tool will produce the desired grid from the given lines, and then this grid will be the input control grid of the standard interpolation method of the B-spline surface.

Here we have to introduce a very effective tool for handling spatial lines: the Plücker coordinates. Let us briefly overview this classical part of geometry.

Consider the three dimensional projective space P^3 . Here every point has four homogeneous coordinates (x_1, x_2, x_3, x_4) , which correlate the Cartesian coordinates (x, y, z) as $x = x_1/x_4, y = x_2/x_4, z = x_3/x_4$ if the point is not at infinity. Now consider a line l of P^3 passing two points (x_1, x_2, x_3, x_4) and (y_1, y_2, y_3, y_4) .

The six Plücker coordinates of this line will be

$$\mathbf{L} = (l_1, \dots, l_6) := (l_{41}, l_{42}, l_{43}, l_{23}, l_{31}, l_{12}), \quad l_{ij} = x_i y_j - x_j y_i.$$

These coordinates do not depend on the choice of the two points on the line l and fulfill the Plücker identity:

$$l_1 l_4 + l_2 l_5 + l_3 l_6 = 0.$$

We consider the four vectors

$$\begin{aligned} \mathbf{s}_0 &= (0, l_{41}, l_{42}, l_{43}), \\ \mathbf{s}_1 &= (-l_{41}, 0, l_{12}, -l_{31}), \\ \mathbf{s}_2 &= (-l_{42}, -l_{12}, 0, l_{23}), \\ \mathbf{s}_3 &= (-l_{43}, l_{31}, -l_{23}, 0). \end{aligned}$$

The four points $\mathbf{s}_0, \dots, \mathbf{s}_3$ are the points where the line with Plücker coordinate vector \mathbf{L} intersects the coordinate planes $x_0 = 0, \dots, x_3 = 0$ (see [13]). For example the intersection point of the line l and the plane $x = 0$ (which is a $[y, z]$ coordinate plane) is $M \left(0, \frac{l_{12}}{-l_{41}}, \frac{l_{31}}{l_{41}} \right)$, if $l_{41} \neq 0$. We can use this results for the reconstruction of the line segments points in the Cartesian coordinates system.

This 1-1 correspondence between the homogeneous 6-tuples of real numbers and the lines of P^3 yields another 1-1 correspondence between the lines of P^3 and the points of the five dimensional projective space P^5 . With the help of this latter mapping we can embed the given rulings and the lines of the future grid of the surface. This way the Kohonen network will be trained by points from P^5 , that is the input layer will contain 6 neurons, while the output layer will be a polygon in P^5 . When the neural network is trained, the result polygon passes through the given points of P^5 . Transforming this situation back to P^3 the given lines and a quadrilateral grid will be received, which grid contains the given lines and consists of straight lines at that direction.

We can examine some working phases of the algorithm in Fig. 1. Fig. 2 shows the final result lines, which is interpolated by B-spline ruled surface. Other running results is shown in Fig. 3.

Whitney’s Umbrella (see Fig. 4) is named after Hassler Whitney, who was one of the scientists founded the differential topology. The equation of the surface is $x^2 - y^2 z = 0$.

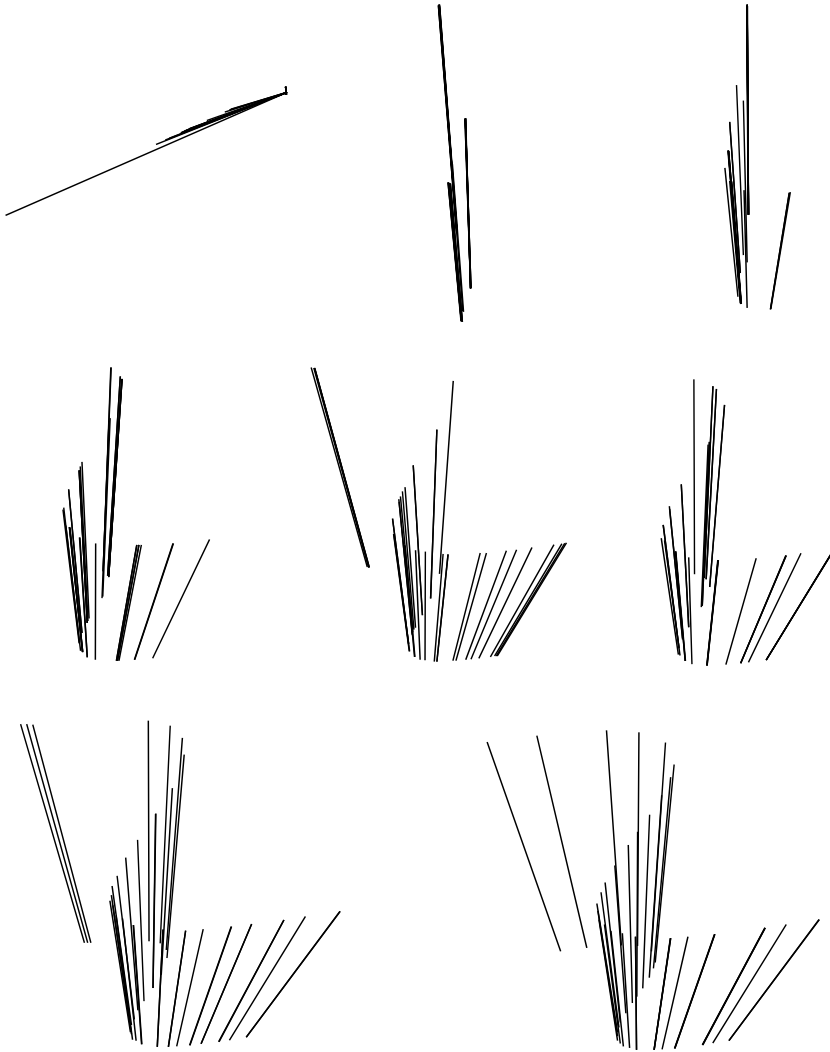


Figure 1: Running Phases

3. The algorithm

Let a set of lines l^i , ($i = 1, \dots, n$) be given. Compute the Plücker coordinates of these lines:

$$l^i(l_j^i) \quad i = 1, \dots, n \quad j = 1, \dots, 6$$

Consider a Kohonen neural network with 6 input neuron and $m (= 4n)$ output neuron. Denote the weights of the connection from the j^{th} input neuron to the i^{th}

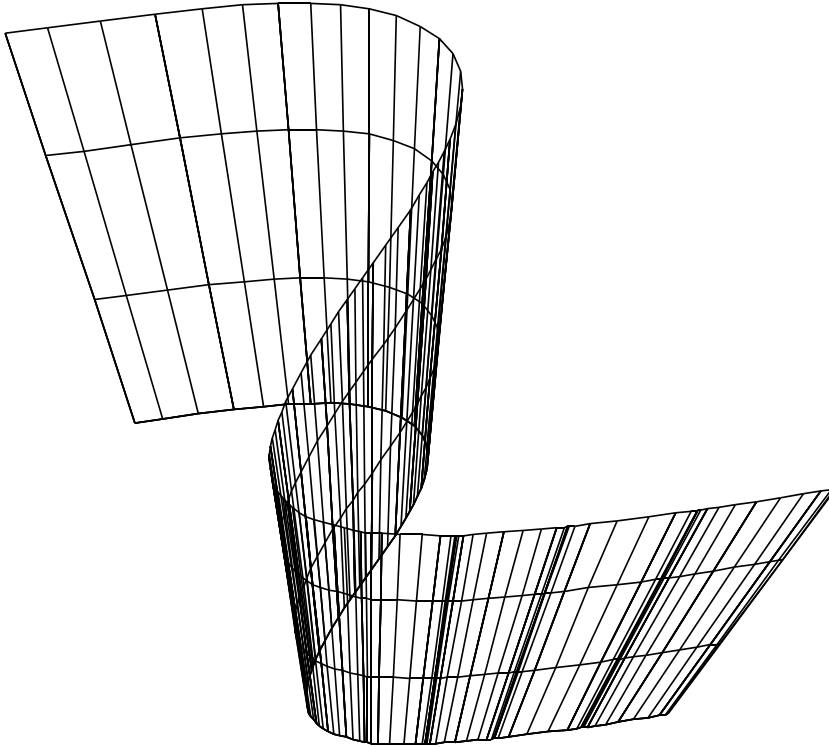


Figure 2: Line segments interpolation by B-spline surface after the training session

output neuron by w_{ij} . Thus the weights $(w_{i0j}), (j = 1, \dots, 6)$ of the connections to an output neuron can be considered as the coordinates of an output point V_i in P^5 .

Now we train the Kohonen neural network. Initializing the weights w_{ij} as small random values around the average of the coordinates of the input points and setting the training time $t = 1$, the coordinates $(l_j^{i_0}), (j = 1, \dots, 6)$ of a randomly chosen input point is presented. The Euclidean distance of all output points to this input point is computed:

$$d_m = \sum_{s=1}^6 (l_s^{i_0} - w_{ms})^2$$

Finding the winning unit, that is the output point closest to the input one, the weights of the connections to this point and to its neighbors are updated by

$$w_{ij}(t + 1) = w_{ij}(t) + \eta(t)(l_j^{i_0} - w_{ij}(t))$$

where t denotes the training time. After updating the weights a new randomly

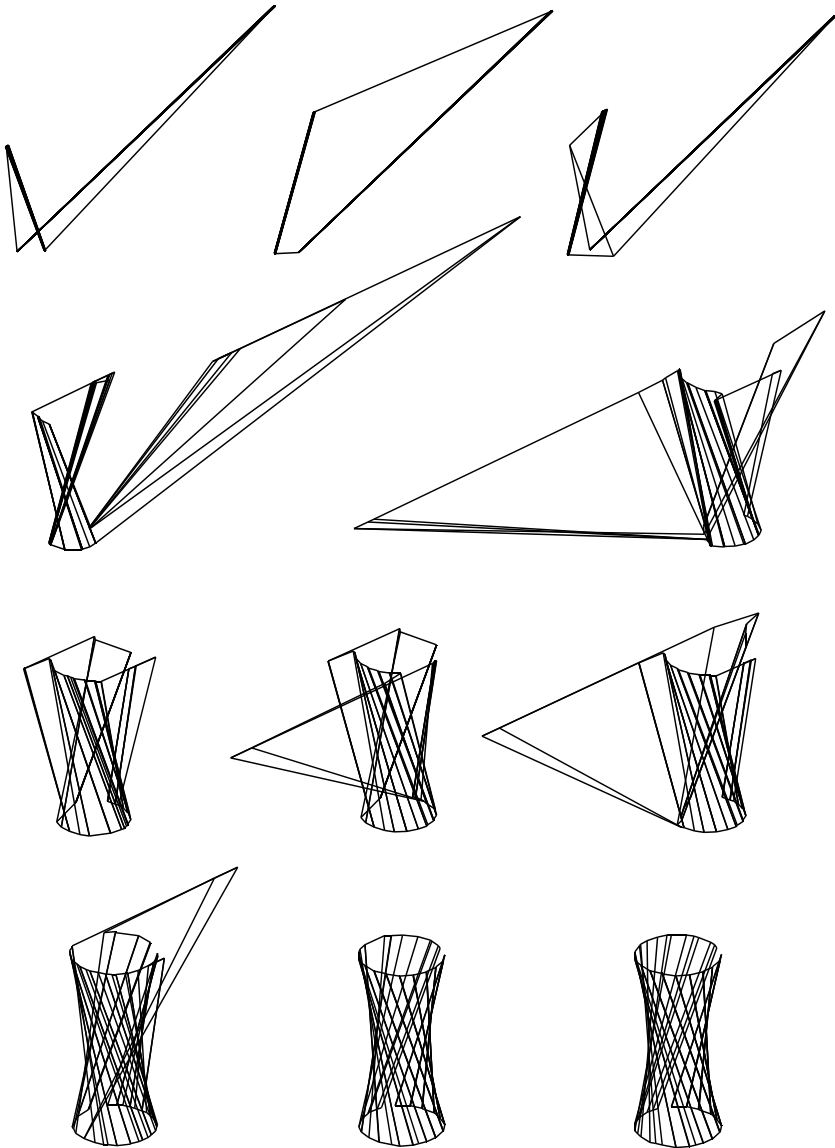


Figure 3: One-sheeted hyperboloid construction by Kohonen network

selected input is presented until the network is trained. The network is said to be trained, if all the input lines are on the grid. If the input structure consists of hundreds of lines, then this requirement would yield long computing time, hence in this case the network is trained if the changes of the weights fall under a predefined

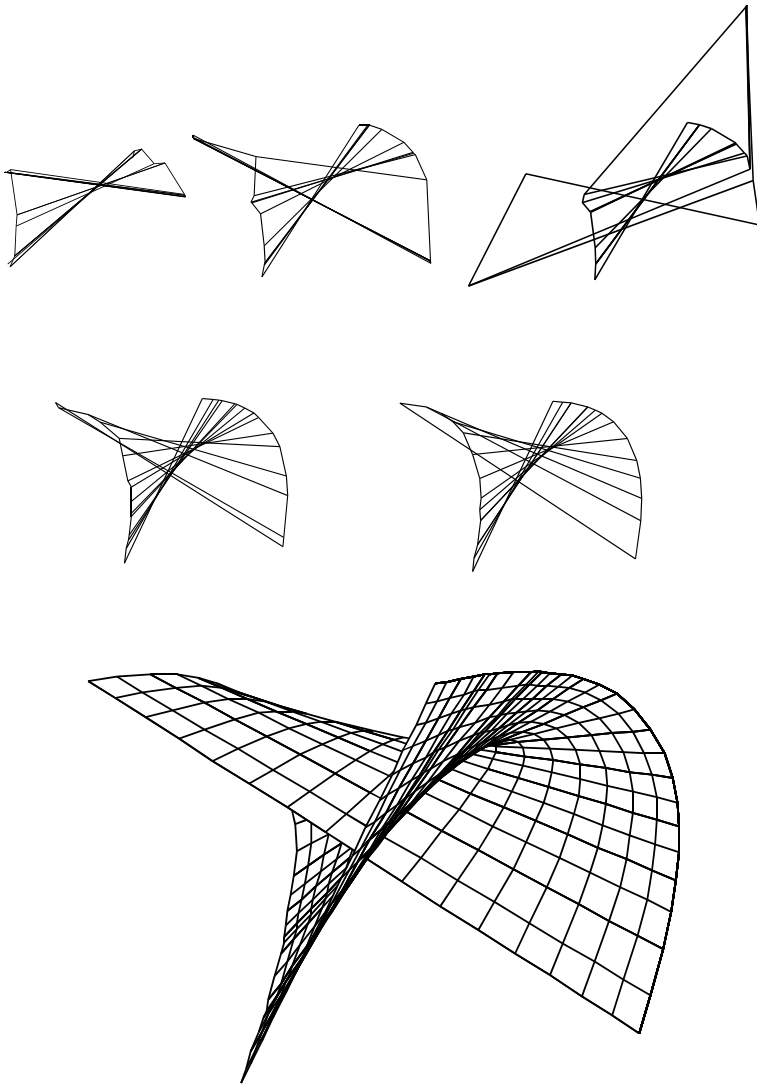


Figure 4: Whitney's Umbrella

limit.

When the training process finished, the result is a polygon in P^5 with the vertices $V_i(w_{ij}), (i = 1, \dots, m), (j = 1, \dots, 6)$. Transforming these vertices back to P^3 the projective lines v_i are received, with the Plücker coordinates (w_{ij}) . Using the definition of these coordinates backwards, the points of these lines can be computed. Thus a set of lines is gained among which the all the original input

lines can be found.

During the training session one can compute the whole grid at every training time t , using the points of the lines v_i as vertices connected also in the other direction. Hence after the training procedure beside the rulings a quadrilateral grid containing these rulings is also received. The vertices of this grid can be applied as input control points to any of the standard free-form surface algorithm. Hence we can interpolate or approximate these points and the rulings.

4. Remarks

Obviously there are infinitely many ruled surface passing through a set of given spatial rulings, and probably each of the methods would produce a different one. Comparing our method with other algorithms the main advantage is the following: beside the self-organizing ability of the Kohonen network it also has a minimal energy property, that is the network tries to minimize the strain energy during the training session. Hence in a sense the result grid is the “smoothest” grid containing all the given rulings, while other methods cannot guarantee this property.

References

- [1] BODDULURI, R., RAVANI, B., Geometric Design and Fabrication of Developable Surfaces, ASME Adv. Design Autom. 2, pp. 243-250, 1992.
- [2] CHEN, H., POTTMANN, H., Approximation by Ruled Surfaces, Journal of Computational and Applied Mathematics, 102, pp. 143-156, 1999.
- [3] FARIN, G., Curves and Surfaces for Computer Aided Geometric Design A Practical Guide, Academic Press, 1996.
- [4] FREEMAN, J., SKAPURA., D., Neural Networks; Algorithms, Applications and Programming Techniques, Addison-Wesley, 1991.
- [5] HLAVATY, V., Differential line geometry, Nordhoff Ltd, Groningen, 1953.
- [6] HOFFMANN M., KOVÁCS E., Constructing ruled B-spline surfaces by Kohonen neural network, Acta Acad. Paed. Agriensis, TOM. XXVII., Sectio Mathematicae, pp. 63-68, 2000.
- [7] HOFFMANN, M., VÁRADY, L., Free-form Surfaces for Scattered Data by Neural Networks, Journal for Geometry and Graphics, Vol.2, No.1, pp. 1-6, 1998.
- [8] HOFFMANN, M., Modified Kohonen Neural Network for Surface Reconstruction, Publ. Math., 54 Suppl., 857-864, 1999
- [9] HOFFMANN, M., Numerical control of Kohonen neural networks for scattered data approximation, Numerical Algorithms, (to appear)

- [10] HOSCHEK, J., SCHWANECKE, U., Interpolation and approximation with ruled surfaces, in: *The Mathematics of Surfaces VII.* (ed.:Robert Cripps), Elsevier, pp. 213–231, 1988.
- [11] KOHONEN, T., *Self-organization and associative memory*, Springer Verlag, 1984.
- [12] KOVÁCS E., *Constructing ruled B-spline surfaces by neural networks, computer graphics relations*, Theses of PhD dissertation 2003
- [13] POTTMANN, H., WALLNER, J., *Computational Line Geometry*, Springer - Verlag, Berlin Heidelberg, 2001.

Postal address

Emőd Kovács

Institute of Mathematics and Informatics

Károly Eszterházy College

Eszterházy tér 1.

3300 Eger, Hungary