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Free-form curve design by knot alteration

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Abstract

Free-form curves, such as B-spline and NURBS curves are defined in a piecewise way over the domain of definition. The section points of the domain are called knots. This paper is devoted to the geometrical and analytical description of curve modification when one or more knot values are altered. Based on the theoretical results some practical shape modification tools are also discussed.

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1. Introduction

Computer aided design and manufacture softwares use free-form curves, especially B-spline curves as standard design tools. One of the main advantages of B-spline curve is the local modification possibility. Due to the local support of the basis functions these curves are defined piecewisely over the domain of definition. The section points of the domain are called knot values. The precise definition is as follows:

Definition 1.1. The recursive function $N_j^k(u)$ given by the equations

$$\begin{array}{l} N_j^1(u) = \begin{cases} 1 & \text{if } u_j \le u < u_{j+1}, \\ 0 & \text{otherwise} \end{cases} \\ N_j^k(u) = \frac{u-u_j}{u_{j+k-1}-u_j} N_j^{k-1}(u) + \frac{u_{j+k}-u}{u_{j+k}-u_{j+1}} N_{j+1}^{k-1}(u) \end{array} \quad (1)$$

is called normalized B-spline basis function of order $1 < k$ (degree $k-1$). The numbers $u_j \leq u_{j+1} \in \mathbf{R}$ are called knot values or simply knots, and $0/0 \doteq 0$ by definition.

Definition 1.2. The curve $\mathbf{s}(u)$ defined by

$$\mathbf{s}(u) = \sum_{l=0}^{n} N_l^k(u)\,\mathbf{d}_l\,, u \in [u_{k-1}, u_{n+1}] \tag{2}$$

is called B-spline curve of order k (degree $k-1$), $(1 < k \leq n+1)$, where $N_l^k(u)$ is the l^{th} normalized B-spline basis function of order k, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are necessary. Points $\mathbf{d}_l$ are called control points or de Boor points, while the polygon formed by these points is called control polygon.

As one can observe, the k^{th} order curve is uniquely defined by its control points and knot values. In case of rational B-spline curves weights are associated with the control points in addition. The properties of B-spline curves under control point reposition and weight modification are well-known for a while (for an overview see e.g. [9]). Several shape control tools were developed based on these properties: beside the most basic ones one can find constrained shape control methods by repositioning control points [2],[10], by modifying weights [5], [10] and by both [1]. It is a well-known fact that the alteration of the knot values also affects the shape of the curve. The geometric properties of this effect has been described recently by the authors [6], [7], [8]. In the next section we give an overwiev of these results. In Section 3 the results are generalized for rational B-spline curves, while in Section 4 practical shape control methods are presented based on these theoretical results.

2. Geometrical properties of knot alteration

Modifying a knot u_i between its neighbours u_{i-1} and u_{i+1} the basis functions $N_j^k(u)$ and the curve $\mathbf{s}(u)$ will also depend on u_i which will be emphasized by the notation $N_j^k(u, u_i)$ and $\mathbf{s}(u, u_i)$. Actually $\mathbf{s}(u, u_i)$ can be considered as a one-parameter family of B-spline curves with the family parameter $u_i \in [u_{i-1}, u_{i+1})$. In [6] we proved the following theorem.

Theorem 2.1. *The family of the k^{th} order B-spline curves*

$$\mathbf{s}(u, u_i) = \sum_{l=0}^{n} \mathbf{d}_l N_l^k(u, u_i)\,,$$

$u \in [u_{k-1}, u_{n+1}]$, $u_i \in [u_{i-1}, u_{i+1})$, $k > 2$ has an envelope. This envelope is a B-spline curve of order $(k-1)$ and can be written in the form

$$\mathbf{b}(v) = \sum_{l=i-k+1}^{i-1} \mathbf{d}_l N_l^{k-1}(v)\,,\ v \in [v_{i-1}, v_i]\,,$$

where $v_j = \begin{cases} u_j & \text{if } j < i \\ u_{j+1} & \text{if } j \geq i \end{cases}$, *that is the* i^{th} *knot value is removed from the knot vector* (u_j) *of the original curves.*

The envelope touches the elements of the family at the points associated with the actual value of u_i and their first derivatives are also proportional there. In [7] this result has been extended for higher order derivatives:

Theorem 2.2. *The relation between the derivatives of the two curves* $\mathbf{s}(u, u_i)$ *and* $\mathbf{b}(v)$ *at* $u = v = u_i$ *is*

$$\frac{d^r}{dv^r}\mathbf{b}(v)\Big|_{v=u_i} = \frac{k-1-r}{k-1}\frac{d^r}{du^r}\mathbf{s}(u, u_i)\Big|_{u=u_i}, \quad \mathrm{r} \geq 0. \tag{3}$$

Another stream of our research was the description of the movement of a single point of the curve. If a control point, a weight or a knot value is modified, the points of the curve move along special curves called paths. These paths are line segments in the first two cases and have been desribed in [10]. Repositioning a control point these line segments are parallel to the movement of the control point. Modifying a weight of a control point the paths are line segments the endpoint of which is the actual control point. Modifying a knot value these paths are rational curves the degree of which decreases in a well-defined way. More precisely we proved the following property (c.f. [6]).

Theorem 2.3. *Modifying the single multiplicity knot* $u_i \in [u_{i-1}, u_{i+1}]$ *of the* k^{th} *order B-spline curve* $\mathbf{s}(u)$, *the point* $\mathbf{s}_j(\widetilde{u})$, $\widetilde{u} \in [u_j, u_{j+1})$ *moves along the path*

$$\mathbf{s}_j(\widetilde{u}, u_i) = \sum_{l=j-k+1}^{j} \mathbf{d}_l N_l^k(\widetilde{u}, u_i), \; u_i \in [u_{i-1}, u_{i+1}]. \tag{4}$$

These paths are rational curves for $j \in [i-k+1, i+k-2]$, *the degree of which decreases symmetrically from* $k-1$ *to* 1 *as the indices of the arcs get farther from* i, *that is, the paths of the points of the arcs* $\mathbf{s}_{i-l}(u)$ *and* $\mathbf{s}_{i+l-1}(u)$ *are rational curves of degree* $k-l$ *in* u_i $(l = 1, \ldots, k-1)$.

Since the degree of these paths decreases symmetrically, an obvious corollary of the theorem above is the following statement.

Consequence 2.1. For $l = k-1$ the paths are of degree 1, that is, the points of the arcs $\mathbf{s}_{i-k+1}(u)$ and $\mathbf{s}_{i+k-2}(u)$ move along straight line segments parallel to the sides $\mathbf{d}_{i-k}, \mathbf{d}_{i-k+1}$ and $\mathbf{d}_{i-1}, \mathbf{d}_i$ of the control polygon, respectively.

These theoretical results can be easily extended for knots with higher multiplicity. In the next section we generalize the results for rational curves as well.

3. The rational case

We adopt the following definition of rational B-spline curves:

Definition 3.1. The curve $\mathbf{s}(u)$ in $\mathbf{R}^d$, $(d > 1)$ defined by the formula

$$\mathbf{s}(u) = \sum_{l=0}^{n} w_l \mathbf{d}_l \frac{N_l^k(u)}{\sum_{j=0}^{n} w_j N_j^k(u)}, \quad u \in [u_{k-1}, u_{n+1}]$$

is called rational B-spline (or NURBS) curve of order k (degree $k-1$), where $N_l^k(u)$ is the l^{th} normalized B-spline basis function of order k, for the evaluation of which the knots $u_0, u_1, \ldots, u_{n+k}$ are needed. Points $\mathbf{d}_l \in \mathbf{R}^d$ are called control or de Boor points, and the scalars $w_0, w_1, \ldots, w_n$ $(w_j \geq 0)$ are called weights.

A rational B-spline curve of Definition 3.1 can be produced by projecting the integral B-spline curve, determined by the same knots and the control points

$$\begin{bmatrix} w_0\mathbf{d}_0 \\ w_0 \end{bmatrix}, \begin{bmatrix} w_1\mathbf{d}_1 \\ w_1 \end{bmatrix}, \ldots, \begin{bmatrix} w_n\mathbf{d}_n \\ w_n \end{bmatrix}$$

in $\mathbf{R}^{d+1}$, from the origin onto the hyperplane $w = 1$.

During central projection the degree of a curve can not increase, thus Theorem 2.3 is valid for the rational case as well. Consequence 2.1 changes a bit, since central projection takes parallel lines to concurrent lines in general. Consequently, modifying the knot u_i of multiplicity m, paths of points of the arc $\mathbf{s}_{i-k+1}(\tilde{u}, u_i)$ are concurrent lines the base of which is the point with homogeneous coordinates

$$(w_{i-k}\mathbf{d}_{i-k} - w_{i-k+1}\mathbf{d}_{i-k+1}, w_{i-k} - w_{i-k+1}).$$

Analogously, paths of the points of $\mathbf{s}_{i+k+m-3}(\tilde{u}, u_i)$ are concurrent lines with the base $(w_{i+m-1}\mathbf{d}_{i+m-1} - w_{i+m-2}\mathbf{d}_{i+m-2}, w_{i+m-1} - w_{i+m-2})$.

We can not provide the rational counterpart of the general formula of Theorem 2.2, but it is clear that G^1 continuity contact and the coincidence of the osculating planes remains valid, since these are geometric properties of curves that are independent of the representation and are preserved during central projection. However, it has to be examined whether the degree of continuity at the common points increases or the type of continuity changes, since for C^r or G^r continuity of rational curves it is not necessary for the corresponding curves in the pre-image plane to be C^r or G^r. (Necessary and sufficient conditions can be found in [4].)

In the forthcoming paragraphs we examine the type and order of continuity at points of contact. For this purpose consider the rational B-spline curve $\mathbf{s}(u)$ of order $k > 3$, defined by the control points $\mathbf{d}_0, \mathbf{d}_1, \ldots, \mathbf{d}_n$, weights $w_0, w_1, \ldots, w_n$ and knots $u_0, u_1, \ldots, u_{n+k}$. When modifying the knot u_i of single multiplicity we obtain the family of curves

$$\begin{aligned} \mathbf{s}(u, u_i) = \tfrac{\mathbf{P}(u,u_i)}{Q(u,u_i)}, u \in [u_{k-1}, u_{n+1}], u_i \in [u_{i-1}, u_{i+1}) \\ \mathbf{P}(u, u_i) = \sum_{l=0}^{n} w_l \mathbf{d}_l N_l^k(u, u_i), \\ Q(u, u_i) = \sum_{l=0}^{n} w_l N_l^k(u, u_i) \end{aligned} \tag{5}$$

We examine the relation between the elements of this family and the rational B-spline curve of order $k-1$

$$\begin{aligned} \mathbf{b}(v) &= \tfrac{\mathbf{R}(v)}{T(v)}, v \in [u_{i-1}, u_{i+1}] \\ \mathbf{R}(v) &= \sum_{l=i-k+1}^{i-1} w_l \mathbf{d}_l N_l^{k-1}(v) \\ T(v) &= \sum_{l=i-k+1}^{i-1} w_l N_l^{k-1}(v) \end{aligned} \tag{6}$$

defined by the same control points and weights, and by the knot values

$$v_j = \begin{cases} u_j, & \text{if} j < i, \\ u_{j+1}, & \text{otherwise.} \end{cases}$$

The rational curves $\mathbf{s}(u, u_i)$ can be produced as the central projection of integral B-spline curves

$$\mathbf{s}^w(u, u_i) = \begin{bmatrix} \mathbf{P}(u, u_i) \\ Q(u, u_i) \end{bmatrix}$$

determined by the control points

$$\begin{bmatrix} w_0 \mathbf{d}_0 \\ w_0 \end{bmatrix}, \begin{bmatrix} w_1 \mathbf{d}_1 \\ w_1 \end{bmatrix}, \ldots, \begin{bmatrix} w_n \mathbf{d}_n \\ w_n \end{bmatrix}.$$

The pre-image of the rational B-spline curve $\mathbf{b}(v)$ is

$$\mathbf{b}^w(v) = \begin{bmatrix} \mathbf{R}(v) \\ T(v) \end{bmatrix}$$

that can be deduced analogously.

The r^{th} derivative of rational B-spline curves of expressions (5) and (6) can be described by the formulae

$$\frac{d^r}{du^r}\mathbf{s}(u, u_i) =$$

$$\frac{1}{Q(u, u_i)} \left(\frac{d^r}{du^r}\mathbf{P}(u, u_i) - \sum_{j=1}^{r} \binom{r}{j} \frac{d^j}{du^j} Q(u, u_i) \frac{d^{r-j}}{du^{r-j}} \mathbf{s}(u, u_i) \right),$$

$$\frac{d^r}{dv^r}\mathbf{b}(v) = \frac{1}{T(v)} \left(\frac{d^r}{dv^r}\mathbf{R}(v) - \sum_{j=1}^{r} \binom{r}{j} \frac{d^j}{dv^j} T(v) \frac{d^{r-j}}{dv^{r-j}} \mathbf{b}(v) \right)$$

cf. e.g. [9]. On this basis we obtain

$$\dot{\mathbf{s}}(u, u_i) = \frac{1}{Q(u, u_i)} \left(\dot{\mathbf{P}}(u, u_i) - \dot{Q}(u, u_i) \mathbf{s}(u, u_i) \right),$$

$$\ddot{\mathbf{s}}(u, u_i) =$$

$$\frac{1}{Q(u,u_i)}\left(\ddot{\mathbf{P}}(u,u_i)-2\dot{Q}(u,u_i)\,\dot{\mathbf{s}}(u,u_i)-\ddot{Q}(u,u_i)\,\mathbf{s}(u,u_i)\right).$$

In the pre-image space Theorem 2.2 is valid by means of which for the derivatives at $u=u_i$, $v=u_i$ we gain

$$\mathbf{b}(u_i)=\mathbf{s}(u_i,u_i),$$

$$\dot{\mathbf{b}}(u_i)=\frac{k-2}{k-1}\dot{\mathbf{s}}(u_i,u_i),$$

$$\ddot{\mathbf{b}}(u_i)=\frac{1}{Q(u_i,u_i)}\left(\frac{k-3}{k-1}\left(\ddot{\mathbf{P}}(u_i,u_i)-\ddot{Q}(u_i,u_i)\,\mathbf{s}(u_i,u_i)\right)-\right.$$
$$\left.-2\left(\frac{k-2}{k-1}\right)^2\dot{Q}(u_i,u_i)\,\dot{\mathbf{s}}(u_i,u_i)\right).$$

Thus for the curvatures $\kappa_s(u_i)$ of $\mathbf{s}(u,u_i)$ at $u=u_i$ and $\kappa_b(u_i)$ of $\mathbf{b}(v)$ at $v=u_i$ we obtain

$$\kappa_b(u_i)=\frac{(k-1)(k-3)}{(k-2)^2}\kappa_s(u_i).$$

Therefore $\mathbf{b}(v)$ is an envelope of the family of curves $\mathbf{s}(u,u_i)$ but the condition of second order geometric continuity contact does not fulfilled, since their curvatures are different.

4. Shape control methods based on knots

Applying the results of Section 2 we developed constraint-based shape control methods by modifying three consecutive knot values. These methods include shape modification of cubic non-rational B-spline curves passing through a point with a prescribed tangent direction, touching a line at a prescribed point of contact or passing through a point at a prescribed parameter value [8]. These methods use purely knot alteration. Here we will focus on shape control tools of NURBS curves by simultaneous changing of knot values and weights. These results are based on the authors previous work [3].

4.1. NURBS curve passing through a point

As we have mentioned, the modification of the weight w_j of a NURBS curve causes a perspective functional translation of points of the effected arcs, i.e. it pulls/pushes points of the curve toward/away from the control point $\mathbf{d}_j$. If a given point is on one of the line segments of the paths of this perspective change, one can easily compute the new weight value such a way, that the new curve will pass through the given point. This point can be *almost* anywhere in the convex hull, but for $k>3$ these concurrent line segments starting from $\mathbf{d}_j$ do not sweep the entire area of the triangle $\mathbf{d}_{j-1},\mathbf{d}_j,\mathbf{d}_{j+1}$ (see the gray area in Fig. 1.). If the given point is close to the side of the control polygon, the problem can be solved only for

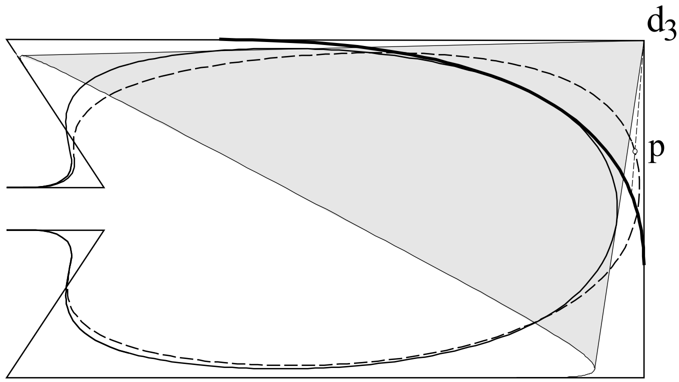

Figure 1: Modifying the weight w_3 and the knot u_4 the NURBS curve passes through a given point $\mathbf{p}$ which is outside the area accessible by modifying purely w_3.

changing two neighbouring weights. Now we give an algorithm solving this problem with the change of one weight and one knot value.

Let a cubic NURBS curve $\mathbf{s}(u)$ and a point $\mathbf{p}$ in the convex hull be given. Let the point $\mathbf{p}$ be in the triangle $\mathbf{d}_{j-1}, \mathbf{d}_j, \mathbf{d}_{j+1}$. Consider the quadratic envelope $\mathbf{b}(v)$ of this NURBS curve changing its knot value u_{j+1}. This parabolic arc intersects all the lines starting from $\mathbf{d}_j$ in this triangle, hence suitably changing the weight w_j there will be a parameter value $\widetilde{v}$, for which $\mathbf{b}(\widetilde{v}) = \mathbf{p}$. Now if we modify the knot value u_{j+1} of the cubic curve for $u_{j+1} = \widetilde{v}$, the cubic curve will also pass through the point $\mathbf{p}$. This type of shape modification is illustrated in Fig. 1.

In this subsection the quadratic envelope has been modified by a weight, where the points of the curve moves along straight lines toward a control point. Similar effect, however, can be achieved in terms of non-rational quadratic B-spline curves by appropriate simultaneous modification of two knot values. More precisely, from the definition of the B-spline functions and Consequence 2.1 one can easily prove the following property:

Theorem 4.1. *The points of the span $\mathbf{s}_{i+1}$ of a non-rational quadratic B-spline curve move along concurrent straight lines with centre $\mathbf{d}_i$, if the knot values u_i and u_{i+3} are changed simultaneously toward (or away from) each other in such a way, that*

$$u_{i+1} - u_i = u_{i+3} - u_{i+2}$$

holds.

As we have seen above, the span $\mathbf{s}_{i+1}$ can be written in the form

$$\mathbf{s}_{i+1}(u) = \mathbf{d}_i + N^3_{i-1}(\mathbf{d}_{i-1} - \mathbf{d}_i) + N^3_{i+1}(\mathbf{d}_{i+1} - \mathbf{d}_i).$$

Consider the path of the point $\mathbf{s}_{i+1}(\widetilde{u})$ and modify u_i and u_{i+3}. Applying the assumption of the theorem we obtain

$$\mathbf{s}_{i+1}(\widetilde{u}, u_i, u_{i+3}) = \mathbf{d}_i + \frac{1}{u_{i+2} - u_i}(C_1(\mathbf{d}_{i-1} - \mathbf{d}_i) + C_2(\mathbf{d}_{i+1} - \mathbf{d}_i))$$

where $C_1$and C_2 are constants. This latter form is an equation of a straight line segment passing through $\mathbf{d}_i$.

The modification of these two knot values, of course, is not so effective, than that of a weight, because the feasible area is greater for the latter case while the number of changing spans is fewer (7 for the two knot values and 3 for the weight), but we have to emphasize, that this theorem allows us to modify non-rational B-spline curves similarly to NURBS curves.

4.2. Modification of two weights and a knot value of a NURBS curve

Modifying two neighbouring weights w_j, w_{j+1} of a NURBS curve the points of the curve move along straight lines toward or away from the leg $\mathbf{d}_j, \mathbf{d}_{j+1}$ of the control polygon. This change is neither perspective nor parallel. This property can be made more intuitive geometrically by modifying a knot value in addition. Thus the points of a span of the curve will move along concurrent lines passing through any given point of the line $\mathbf{d}_j, \mathbf{d}_{j+1}$ except the inner point of the leg. As we have mentioned in the preceding section, modifying a knot value u_j of a cubic NURBS curve the points of the spans $\mathbf{s}_{j-3}, \mathbf{s}_{j+2}$ will move along two families of concurrent straight lines. Considering the span $\mathbf{s}_{j-3}$ and assuming that $w_{j-4} \neq w_{j-3}$ the following result can be achieved: modifying the knot value u_j the points of this span move along concurrent lines the centre of which is on the line $\mathbf{d}_j, \mathbf{d}_{j+1}$ and its barycentric coordinates are

$$\left(\frac{w_{j-4}}{w_{j-4} - w_{j-3}}, 1 - \frac{w_{j-4}}{w_{j-4} - w_{j-3}}\right).$$

We can easily see, that one of its coordinates must be negative with the usual assumption $w_j \geq 0$ for $\forall j$. Hence this centre cannot be on the leg $\mathbf{d}_j, \mathbf{d}_{j+1}$ but on the rest of the line. Fig. 2. shows a case of this type of modification.

5. Conclusion and further research

This paper has been devoted to the shape control of cubic B-spline and NURBS curves. These curves can be uniquely defined by their degree, control points, weights and knot vector and while the effect of the modification of the preceding data has been widely published and used, the change of the knot vector has just been studied in some recent papers of the authors. At the first sections some theoretical results have been presented in terms of the paths of the points of the

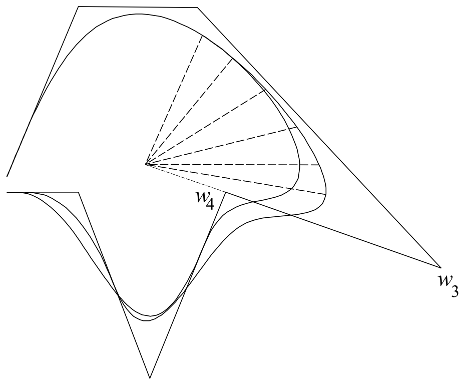

Figure 2: Modifying the knot value u_7 the points of the span $\mathbf{s}_4$ moves along concurrent straight lines the centre of which depends on w_3 and w_4 and can be arbitrary chosen on the line of $\mathbf{d}_3\mathbf{d}_4$.

curve modifying one of its knot value and the existence of an envelope of the resulted family of curves. Applying these results some shape control methods have been presented in the last section for NURBS curves. Simultaneous change of one or more weights and knot values has been presented, the result of which is a NURBS curve passing through a given point or a geometrically simple perspective shape modification.

One of the main stream of our further research will be the theoretical aspects of knot modification for surfaces, which will hopefully generate some shape control methods also for B-spline and NURBS surfaces.

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