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Fundamental D-V cells for $\mathbb{E}^4$ space groups on the 2D-screen

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Abstract

As an introduction we describe two algorithms for determining fundamental domains (D-V cells) of **XIX.27/01/01** and **XXII.31/05/02/002** [BBNWZ] space groups. With the first algorithm we obtain a 4-polytope as fundamental domain of **XIX.27/01/01** space group [AM01] belonging to the decagonal crystal family. The second algorithm determines a fundamental domain of a space group with broken translations [AM02].

Then we describe a method for orthogonal projection of $\mathbb{E}^d$-pictures on **2D**-screen, and we can illustrate the edge structure of D-V cells for 25 space groups to the decagonal and icosahedral families of $\mathbb{E}^4$.

Key Words: crystallographic group, four-dimensional space group, decagonal family, icosahedral family, fundamental domain, Dirichlet-Voronoi cell

MSC 2000: 20H15, 51F15

1. Description of Space Groups in $\mathbb{E}^d$

A *space group* Γ is defined in space $\mathbb{E}^d$ as a group of isometries with a *compact* (bounded and closed) *fundamental domain* $\mathcal{F}$ i.e.

$$\bigcup_{\gamma\in\Gamma} \gamma\mathcal{F} = \mathbb{E}^d \quad \text{and} \quad \text{Int}\mathcal{F} \cap \text{Int}\gamma\mathcal{F} = \emptyset \quad \text{for any} \quad \gamma \in \Gamma \setminus \{\mathbf{1}\}$$

("Int" abreviates interior, $\gamma\mathcal{F}$ is the γ-image of $\mathcal{F}$, $\mathbf{1}$ denotes the identity map). Any element α of Γ associates each point X with its image $\alpha X =: Y$ by $(d+1)$-row-column multiplication as usual:

$$\begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix} := \begin{pmatrix} \mathbf{A} & \mathbf{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{x}+\mathbf{a} \\ 1 \end{pmatrix}; X\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix}, Y\begin{pmatrix} \mathbf{y} \\ 1 \end{pmatrix}$$

are introduced. We write $\alpha(\mathbf{a};\mathbf{A})$ as well. The $d\times d$ matrix $\mathbf{A}$ is called the *linear part* of α, the column vector $\mathbf{a}$ is its *translational part.* All these are expressed in a coordinate system $(O,\mathbf{e}_1,\dots,\mathbf{e}_d)$ with an origin O, and a so called *lattice basis* $(\mathbf{e}_1,\dots,\mathbf{e}_d)$. $\mathbf{L}$ is the *integral linear combinations* of the basis vectors $(\mathbf{e}_1,\dots,\mathbf{e}_d)$

$$\mathbf{L}=\left\{\mathbf{l}=\mathbf{e}_1 l^1+\dots+\mathbf{e}_d l^d \,:\, (l^1,\dots,l^d)\in\mathbb{Z}^d\right\},$$

where $\mathbb{Z}$ denotes the set of integers.

1.1. D-V Cell and Fundamental Domain

The D-V *cell of the kernel point* P to its Γ-orbit is

$$\mathcal{D}_\Gamma(P)=\{X\in\mathbb{E}^d \,:\, \rho(P,X)\le\rho(\gamma P,X) \quad \text{for each} \quad \gamma\in\Gamma\}$$

2. Fundamental domain for space group XIX.27/01/01

The Gramian matrix in the decagonal family is:

$$G_d=\begin{pmatrix} a & b & -\frac{1}{2}(a+2b) & -\frac{1}{2}(a+2b)\\ b & a & b & -\frac{1}{2}(a+2b)\\ -\frac{1}{2}(a+2b) & b & a & b\\ -\frac{1}{2}(a+2b) & -\frac{1}{2}(a+2b) & b & a \end{pmatrix}.$$

To simplify our discussion, $b=-\frac{1}{4}a$ will be assumed, since we want to find a fundamental domain as simple as possible. The point group Γ_{05} of space group $27/01/01=\Gamma_5$ has one generator: $\gamma_{\mathbf{5}}$, mapping

$$\mathbf{e_1}\to\mathbf{e_4}\to\mathbf{e_2}\to-\mathbf{e_1}-\mathbf{e_2}-\mathbf{e_3}-\mathbf{e_4}\to\mathbf{e_3}\to\mathbf{e_1}$$

cyclically, thus $\gamma_{\mathbf{5}}$ is a transform of order five, indeed.

2.1. Algorithm for the fundamental domain $\mathcal{F}_{\mathbf{5}}$ of space group $\mathbf{\Gamma_5}$

First: We choose the kernel point $P_1\,(1;1;1;1)$, and determine the *3-plane bisectors of* P_1 *and* γP_1, $\gamma\in\Gamma_{05}$ (P_1 lies in negative halfspaces of the former *3-planes*).

The transforms and the equations of 3-planes will be

$$\gamma_{\mathbf{5}}: \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -z=0; \quad \gamma_{\mathbf{5}}^{\mathbf{2}}: \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -x=0;$$

$$\gamma_{\mathbf{5}}^{\mathbf{3}}: \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -w=0; \quad \gamma_{\mathbf{5}}^{\mathbf{4}}: \begin{pmatrix} 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, -y=0.$$

Thus we get a pyramidal domain $\mathcal{F}_{05}$ with apex O that will contain $\mathcal{F}_5$.
Second: We determine the equations of *bisector 3-planes of the origin O and its Γ_5-images, now the lattice points with $0, 1, -1$ coordinates* . In case of Γ_5 it will be enough to take points with $0, 1$ coordinates. The origin O always lies in negative halfspaces of the above 3-planes. Thus we can intersect the pyramid by new 3-planes
The lattice points and the equations of 3-planes:

$$\begin{aligned}
&(1;1;1;1;1)^T, x+y+z+w-2=0\\
&(1;0;0;0;1)^T, 4x-y-z-w-2=0,\\
&(0;1;0;0;1)^T, -x+4y-z-w-2=0,\\
&(0;0;1;0;1)^T, -x-y+4z-w-2=0,\\
&(0;0;0;1;1)^T - x-y-z+4w-2=0,\\
&(1;1;0;0;1)^T, 3x+3y-2z-2w-3=0,\\
&(1;0;1;0;1)^T, 3x-2y+3z-2w-3=0,\\
&(1;0;0;1;1)^T, 3x-2y-2z+3w-3=0,\\
&(0;1;1;0;1)^T, -2x+3y+3z-2w-3=0,\\
&(0;1;0;1;1)^T, -2x+3y-2z+3w-3=0,\\
&(0;0;1;1;1)^T, -2x-2y+3z+3w-3=0,\\
&(1;1;1;0;1)^T, 2x+2y+2z-3w-3=0,\\
&(1;1;0;1;1)^T, 2x+2y-3z+2w-3=0,\\
&(1;0;1;1;1)^T, 2x-3y+2z+2w-3=0,\\
&(0;1;1;1;1)^T, -3x+2y+2z+2w-3=0.
\end{aligned}$$

Third: We start with five suitable equations of 3-planes and determine the *5 vertices* of a starting simplex (*5-cell*). We take a new 3-plane, substitute the coordinates of all vertices of the convex *4-dimensional polyhedron* into its equation. If at least one vertex exists in the positive halfspace of the 3-plane, then we cut the polyhedron, otherwise leave it.

Finally, the fundamental domain $\mathcal{F}_5$ has **19** *geometric 3-faces and* **65** *proper vertices.*

Fundamental domain of space group 27/01/01 ($\mathcal{F}_5$)

Element generator

$$\gamma_5 = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

3-faces

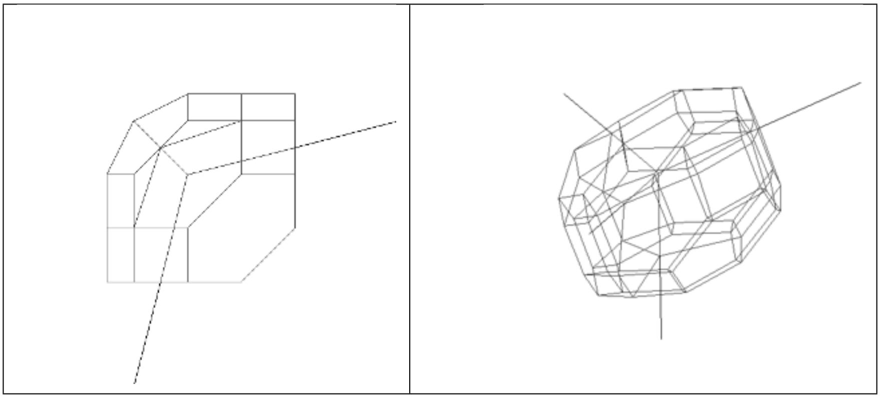
$-x=0,\quad -y=0,\quad -z=0,\quad -w=0,$

$x+y+z+w-2=0,$	$4x-y-z-w-2=0,$	$-x+4y-z-w-2=0,$
$-x-y+4z-w-2=0,$	$-x-y-z+4w-2=0,$	$3x+3y-2z-2w-3=0,$
$3x-2y+3z-2w-3=0,$	$3x-2y-2z+3w-3=0,$	$-2x+3y+3z-2w-3=0,$
$-2x+3y-2z+3w-3=0,$	$-2x-2y+3z+3w-3=0,$	$2x+2y+2z-3w-3=0,$
$2x+2y-3z+2w-3=0,$	$2x-3y+2z+2w-3=0,$	$-3x+2y+2z+2w-3=0;$

Vertices

$\{(\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5}),$	$(\frac{4}{5},\frac{3}{5},\frac{1}{5},\frac{2}{5}),$	$(\frac{4}{5},\frac{2}{5},\frac{3}{5},\frac{1}{5}),$	$(\frac{4}{5},\frac{2}{5},\frac{1}{5},\frac{3}{5}),$	$(\frac{4}{5},\frac{1}{5},\frac{3}{5},\frac{2}{5}),$
$(\frac{4}{5},\frac{1}{5},\frac{2}{5},\frac{3}{5}),$	$(\frac{3}{5},\frac{4}{5},\frac{2}{5},\frac{1}{5}),$	$(\frac{3}{5},\frac{4}{5},\frac{1}{5},\frac{2}{5}),$	$(\frac{3}{5},\frac{2}{5},\frac{4}{5},\frac{1}{5}),$	$(\frac{3}{5},\frac{2}{5},\frac{1}{5},\frac{4}{5}),$
$(\frac{3}{5},\frac{1}{5},\frac{4}{5},\frac{2}{5}),$	$(\frac{3}{5},\frac{1}{5},\frac{2}{5},\frac{4}{5}),$	$(\frac{2}{5},\frac{4}{5},\frac{3}{5},\frac{1}{5}),$	$(\frac{2}{5},\frac{4}{5},\frac{1}{5},\frac{3}{5}),$	$(\frac{2}{5},\frac{3}{5},\frac{4}{5},\frac{1}{5}),$
$(\frac{2}{5},\frac{3}{5},\frac{1}{5},\frac{4}{5}),$	$(\frac{2}{5},\frac{1}{5},\frac{4}{5},\frac{3}{5}),$	$(\frac{2}{5},\frac{1}{5},\frac{3}{5},\frac{4}{5}),$	$(\frac{1}{5},\frac{4}{5},\frac{3}{5},\frac{2}{5}),$	$(\frac{1}{5},\frac{4}{5},\frac{2}{5},\frac{3}{5}),$
$(\frac{1}{5},\frac{3}{5},\frac{4}{5},\frac{2}{5}),$	$(\frac{1}{5},\frac{3}{5},\frac{2}{5},\frac{4}{5}),$	$(\frac{1}{5},\frac{2}{5},\frac{4}{5},\frac{3}{5}),$	$(\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}),$	$(\frac{7}{10},\frac{1}{2},\frac{3}{10},0),$
$(\frac{7}{10},\frac{1}{2},0,\frac{3}{10}),$	$(\frac{7}{10},\frac{3}{10},\frac{1}{2},0),$	$(\frac{7}{10},\frac{3}{10},0,\frac{1}{2}),$	$(\frac{7}{10},0,\frac{1}{2},\frac{3}{10}),$	$(\frac{7}{10},0,\frac{3}{10},\frac{1}{2}),$
$(\frac{1}{2},\frac{7}{10},\frac{3}{10},0),$	$(\frac{1}{2},\frac{7}{10},0,\frac{3}{10}),$	$(\frac{1}{2},\frac{3}{10},\frac{7}{10},0),$	$(\frac{1}{2},\frac{3}{10},0,\frac{7}{10}),$	$(\frac{1}{2},0,\frac{7}{10},\frac{3}{10}),$
$(\frac{1}{2},0,\frac{3}{10},\frac{7}{10}),$	$(\frac{3}{10},\frac{7}{10},\frac{1}{2},0),$	$(\frac{3}{10},\frac{7}{10},0,\frac{1}{2}),$	$(\frac{3}{10},\frac{1}{2},\frac{7}{10},0),$	$(\frac{3}{10},\frac{1}{2},0,\frac{7}{10}),$
$(\frac{3}{10},0,\frac{7}{10},\frac{1}{2}),$	$(\frac{3}{10},0,\frac{1}{2},\frac{7}{10}),$	$(0,\frac{7}{10},\frac{1}{2},\frac{3}{10}),$	$(0,\frac{7}{10},\frac{3}{10},\frac{1}{2}),$	$(0,\frac{1}{2},\frac{7}{10},\frac{3}{10}),$
$(0,\frac{1}{2},\frac{3}{10},\frac{7}{10}),$	$(0,\frac{3}{10},\frac{7}{10},\frac{1}{2}),$	$(0,\frac{3}{10},\frac{1}{2},\frac{7}{10}),$	$(\frac{3}{5},\frac{2}{5},0,0),$	$(\frac{3}{5},0,\frac{2}{5},0),$
$(\frac{3}{5},0,0,\frac{2}{5}),$	$(\frac{2}{5},\frac{3}{5},0,0),$	$(\frac{2}{5},0,\frac{3}{5},0),$	$(\frac{2}{5},0,0,\frac{3}{5}),$	$(0,\frac{3}{5},\frac{2}{5},0),$
$(0,\frac{3}{5},0,\frac{2}{5}),$	$(0,\frac{2}{5},\frac{3}{5},0),$	$(0,\frac{2}{5},0,\frac{3}{5}),$	$(0,0,\frac{3}{5},\frac{2}{5}),$	$(0,0,\frac{2}{5},\frac{3}{5}),$
$(\frac{1}{2},0,0,0),$	$(0,\frac{1}{2},0,0),$	$(0,0,\frac{1}{2},0),$	$(0,0,0,\frac{1}{2}),$	$(0,0,0,0)\};$

The edgestructure of $\mathcal{F}_5$



3. Fundamental domain ($\mathcal{F}_{\mathbf{05t}}$) for space group XXII.31/05/02/002

The icosahedral crystal family has two Bravais lattices with Gramians

$$\begin{pmatrix} a & -\frac{a}{4} & -\frac{a}{4} & -\frac{a}{4} \\ -\frac{a}{4} & a & -\frac{a}{4} & -\frac{a}{4} \\ -\frac{a}{4} & -\frac{a}{4} & a & -\frac{a}{4} \\ -\frac{a}{4} & -\frac{a}{4} & -\frac{a}{4} & a \end{pmatrix}, \begin{pmatrix} a & \frac{a}{2} & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & a & \frac{a}{2} & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & a & \frac{a}{2} \\ \frac{a}{2} & \frac{a}{2} & \frac{a}{2} & a \end{pmatrix},$$

for the primitive and its SN centred (seitenflächen-nebendiagonal-zentriert) lattices.

XXII.31/05/02/002 $= \Gamma_{5t}$ is the richest space group for the SN centred lattice with broken translations [BBNWZ]. The 120 transform of its point group Γ_{05t} and all transforms of Γ_{5t} itself can be generated by homogeneous 5x5 matrices, as usual

$$\gamma_{4t} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{2}{3} \\ 0 & -1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \gamma_{5t} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_{6t} = \begin{pmatrix} 0 & 0 & -1 & 0 & \frac{1}{3} \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3.1. Algorithm for the fundamental domain $\mathcal{F}_{\mathbf{5t}}$ of space group $\mathbf{\Gamma_{5t}}$

First: We determine all broken translations of Γ_{5t} in increasing distances, say, to the origin O, at the same time we obtain the transforms without translations $\gamma_{05} \in \Gamma_{stab(O)} < \Gamma_{05t}$.

Second: We determine the equations of *bisector 3-planes of the origin O and its broken translates.*

Third: We choose the second kernel point $P_1\,(1;1;1;1)$, and determine the pyramidal domain $\mathcal{F}_{Stab(O)}$ with γ_{05}'s fixing O.

Fourth: We apply the algorithm in the third point of previous chapter, and we obtain the equations of 3-planes and the vertices of the fundamental domain.

Our fundamental domain $\mathcal{F}_{5t}$ will have **20** *bisector 3-faces and* ***45*** *proper vertices.*

Fundamental domain of space group 31/05/02/002 ($\mathcal{F}_{5t}$)

Generator elements

$$\gamma_{4t} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 & \frac{2}{3} \\ 0 & -1 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & -1 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma_{5t} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & \frac{2}{3} \\ 1 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\gamma_{6t} = \begin{pmatrix} 0 & 0 & -1 & 0 & \frac{1}{3} \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix};$$

4. $\mathbb{E}^{\mathbf{d}}$-pictures on 2-screen

Let $(\mathbf{a_1}, \mathbf{a_2}, ...\mathbf{a_d}, \mathbf{a_0})$ be a fixed homogeneous coordinate system of $\mathbb{E}^d \subset \mathcal{P}^d$ with given (possibly skew)Gramian

$$(< \mathbf{a}_i, \mathbf{a}_j >); \quad \mathbf{a}_{ij} = \mathbf{a}_{ji}, \quad \text{for} \quad i, j = 1, \dots, d.$$

Let $(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_0(t))$ be Cartesian homogeneous coordinate system of the screen, moving by the time t, expressed in $\mathbf{a}_i$:

$$\mathbf{e}_\alpha(t) = \mathbf{a}_i \mathbf{e}^i_\alpha(t), \quad \alpha = 0, 1, 2; \quad i = 0, 1, \dots, d;$$

with the projective freedom of non-zero $\mathbf{R}$-factor, and the orthogonality

$$e_{\alpha\beta} :=< \mathbf{e}_\alpha, \mathbf{e}_\beta >=< \mathbf{a}_r e^r_\alpha, \mathbf{a}_s e^s_\beta >= e^r_\alpha a_{rs} e^s_\beta = \delta_{\alpha\beta}$$
$$\alpha, \beta \in \{1, 2\}; \quad r, s \in \{1, 2, \dots .d\}.$$

Any point $X(\mathbf{x} = \mathbf{a}_i x^i)$ will be orthogonally projected onto the screen

$$(\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_0(t)),$$

and the image $Y(t)(\mathbf{y}(t) = \mathbf{e}_\alpha(t) y^\alpha(t))$ of X is to express by (x^i), (a_{rs}), $e^k_\alpha(t)$.

The crucial assumption is

$$\overrightarrow{XY} \quad \perp \quad (\mathbf{e}_1(t), \mathbf{e}_2(t), \mathbf{e}_0(t)) \quad \text{i.e.}$$
$$< \mathbf{a}_i e^i_\alpha(t) y^\alpha(t) - \mathbf{a}_i x^i c(t), \mathbf{a}_k e^k_\beta(t) - \mathbf{a}_k e^k_0(t) d_\beta(t) >= 0$$

Here the “functions” $c(t)$ and $d_\beta(t)$ are determined by

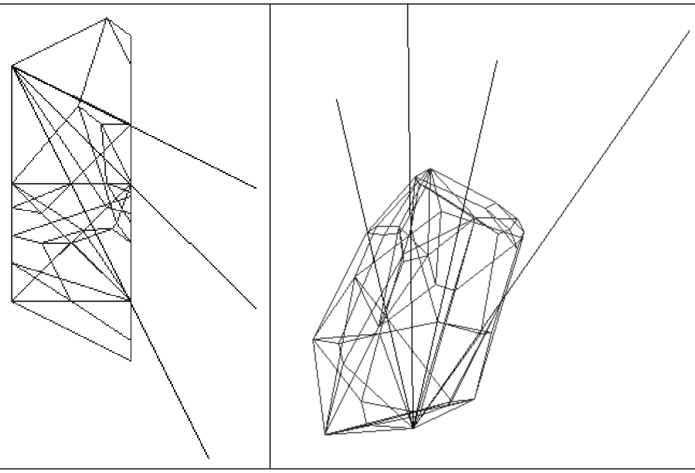
$$e^0_\alpha(t) y^\alpha(t) = x^0 c(t),$$
and
$$e^0_\beta(t) = e^0_0(t) d_\beta(t),$$

in general, but $x^0 = 1$, $e_0^0(t) = 1, e_1^0 = e_2^0 = 0$, thus $y^0 = 1$, $c(t) = 1$, $d_0(t) = 1, d_1(t) = d_2(t) = 0$ can be assumed from the beginning for any time t later. The cases $x^0 = 0$ lead $y^0 = 0$, i.e. ideal points have ideal images (at the infinity).

3-faces		
$-x-y-z=0,$	$-x-y-w=0,$	$-y-z-w=0,$
$-2x-y-z-w=0,$	$-x-y-2z-w=0,$	$-x-y-z-2w=0,$
$3z-3w-1=0,$	$-3z+3w-1=0,$	$6x+3y+3z+3w-1=0,$
$6x+6y+3z-2=0,$	$6x+3z+6w-2=0,$	$3x+6y+6z-2=0,$
$-3x+3y-3z+3w-2=0,$	$9x+3y+9z+9w-4=0,$	$6y+9w-4=0,$
$6x+3y+9z+12w-5=0,$	$3x+12y+9z+6w-5=0,$	$3x+9y+12z+6w-5=0,$
$-3x+9y+3z+6w-5=0,$	$-3x+6y+3z+9w-5=0;$	

Vertices			
$\{(\frac{-14}{45},\frac{11}{45},\frac{11}{45},\frac{2}{15}),$	$(\frac{-14}{45},\frac{11}{45},\frac{2}{15},\frac{11}{45}),$	$(\frac{-1}{13},\frac{-2}{39},\frac{16}{39},\frac{5}{39}),$	$(\frac{-5}{39},\frac{2}{39},\frac{1}{13},\frac{16}{39}),$
$(\frac{-2}{27},\frac{-1}{27},\frac{11}{27},\frac{1}{9}),$	$(\frac{-1}{9},\frac{1}{27},\frac{2}{27},\frac{11}{27}),$	$(\frac{-1}{9},\frac{-5}{27},\frac{8}{27},\frac{8}{27}),$	$(\frac{-8}{27},\frac{5}{27},\frac{8}{27},\frac{1}{9}),$
$(\frac{-8}{27},\frac{5}{27},\frac{1}{9},\frac{8}{27}),$	$(\frac{-1}{15},\frac{13}{30},\frac{-1}{15},\frac{1}{10}),$	$(\frac{-7}{30},\frac{4}{15},\frac{1}{10},\frac{4}{15}),$	$(\frac{2}{15},\frac{4}{15},\frac{-2}{15},\frac{-2}{15}),$
$(\frac{2}{15},\frac{2}{15},\frac{2}{15},\frac{-1}{5}),$	$(\frac{2}{15},\frac{-4}{15},\frac{2}{15},\frac{2}{15}),$	$(\frac{1}{15},\frac{2}{5},\frac{-4}{15},\frac{1}{15}),$	$(\frac{-1}{10},\frac{1}{15},\frac{1}{15},\frac{2}{5}),$
$(\frac{-1}{15},\frac{4}{15},\frac{-1}{15},\frac{4}{15}),$	$(\frac{-1}{15},\frac{-1}{15},\frac{4}{15},\frac{4}{15}),$	$(\frac{-2}{15},\frac{8}{15},\frac{-2}{15},\frac{-2}{15}),$	$(\frac{-2}{15},\frac{4}{15},\frac{2}{15},\frac{-2}{15}),$
$(\frac{-2}{15},\frac{4}{15},\frac{-2}{15},\frac{2}{15}),$	$(\frac{-2}{15},\frac{1}{5},\frac{1}{5},\frac{1}{5}),$	$(\frac{-2}{15},\frac{-1}{15},\frac{2}{5},\frac{1}{5}),$	$(\frac{-1}{5},\frac{1}{15},\frac{2}{15},\frac{2}{5}),$
$(\frac{-4}{15},\frac{2}{5},\frac{1}{15},\frac{1}{15}),$	$(\frac{1}{6},\frac{1}{4},\frac{-1}{6},\frac{-1}{12}),$	$(\frac{1}{6},\frac{-1}{4},\frac{1}{6},\frac{1}{12}),$	$(\frac{-1}{12},\frac{1}{4},\frac{-1}{6},\frac{1}{6}),$
$(\frac{-1}{12},\frac{-1}{6},\frac{1}{3},\frac{1}{4}),$	$(\frac{-1}{4},\frac{1}{6},\frac{1}{12},\frac{1}{3}),$	$(\frac{-1}{3},\frac{2}{9},\frac{2}{9},\frac{2}{9}),$	$(\frac{-1}{9},\frac{1}{3},0,\frac{2}{9}),$
$(\frac{-1}{9},0,\frac{1}{3},\frac{2}{9}),$	$(\frac{-1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{3}),$	$(\frac{1}{6},\frac{1}{6},0,\frac{-1}{6}),$	$(\frac{1}{6},0,\frac{1}{6},\frac{-1}{6}),$
$(\frac{1}{6},0,\frac{-1}{6},\frac{1}{6}),$	$(\frac{1}{6},\frac{-1}{6},0,\frac{1}{6}),$	$(0,\frac{1}{6},\frac{1}{6},\frac{-1}{6}),$	$(0,\frac{-1}{6},\frac{1}{3},\frac{1}{6}),$
$(\frac{-1}{6},\frac{1}{6},0,\frac{1}{3}),$	$(\frac{-1}{3},0,\frac{1}{3},\frac{1}{3}),$	$(0,0,\frac{1}{3},0),$	$(0,0,0,\frac{1}{3}),$
$(0,0,0,0)\};$			

The edgestructure of $\mathcal{F}_{5t}$



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