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Some unexpected properties of partial propositional logic based on partial approximation space

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Abstract

Rough set theory can be considered as the foundation of various kinds of deductive reasoning. In this paper, the authors do some logical investigations about the possibility of using partial approximation of sets as semantics of a three-valued partial logic which use one-argument predicate parameters without quantification. As a consequence of using the lower and upper approximation of sets, approximative functors appear in object language. Functors and three-valued semantics give a real possibility to investigate how to alter valid logical laws.

Keywords: approximation of sets, rough set, partial logic, partial semantics

MSC: 03B50

1. Introduction

In recent years, a number of theoretical attempts have appeared in order to approximate sets. For example, rough set theory was originally proposed by Pawlak (see in [1], [2]), its different generalizations (see, e.g. in [3])

In this paper we go on our logical investigations about the possibility of using different systems of set approximation in a quantification-free logical semantics. First we want to show some unexpected semantic properties of approximation, and later we define some necessary condition to avoid the lack of connection betvenn the evaluation results of different approximation.

2. Partial approximation of sets

In the following definition a most fundamental (and very general) notion of an approximation space is given. This core notion serves as the set–theoretical background of semantics of partial logic with approximative operators.

Definition 2.1. The ordered 5–tuple $\langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ is a general partial approximation space if

1. U is a nonempty set;
2. $\mathfrak{B} \subseteq 2^U \setminus \emptyset$, $\mathfrak{B} \neq \emptyset$;
3. $\mathfrak{D}_{\mathfrak{B}}$ is an extension of $\mathfrak{B}$, i.e. $\mathfrak{B} \subseteq \mathfrak{D}_{\mathfrak{B}}$, such that $\emptyset \in \mathfrak{D}_{\mathfrak{B}}$;
4. the functions l, u forms a Pawlakian approximation pair $\langle \mathsf{l}, \mathsf{u}\rangle$, i.e.
   (a) $\mathsf{l}(S) = \cup\{B : B \in \mathfrak{B} \wedge B \subseteq S\}$;
   (b) $\mathsf{u}(S) = \cup\{B : B \in \mathfrak{B} \wedge B \cap S \neq \emptyset\}$.

Informally, the set U is the universe of approximation; $\mathfrak{B}$ is a nonempty set of base sets, it represent our knowledge used in the whole approximation process; $\mathfrak{D}_{\mathfrak{B}}$ (i.e. the set of definable sets) contains not only the base sets, but those which we want to use to approximate any subset of U; the functions l, u determine the lower and upper approximation of any set with the help of representations of our primitive or available concepts/properties. The nature of an approximation pair[1] depends on how to relate the lower and upper approximations of a set to the set itself

3. Partial quantification–free logic

At first we need to give a language of quantification–free logic. For sake of simplicity, we use one-argument predicate parameters. A finite nonempty set $\mathcal{T}$ of them are called tools.

3.1. Language with approximative functors

Definition 3.1. $L = \langle LC, Var, Con, \mathfrak{T}, Form, Snt\rangle$ is a language, if

1. $LC = \{\neg, \supset, +, \downarrow, \uparrow, (,)\}$, LC is the set of logical constants.
2. $Var = \{x_i : i = 0, 1, 2, \dots\}$, Var is the denumerable infinite set of individual variables.
3. For simplicity $Con = \mathcal{P}(1)$, where $\mathcal{P}(1)$ is the set of one-argument predicate parameters.
4. The set of tools $\mathfrak{T}$ is finite, $\mathfrak{T} \subseteq \mathcal{P}(1)$ and $\mathfrak{T} \neq \emptyset$.
5. The sets LC, Var, Con are pairwise disjoint.

[1]One of the most general notion of weak and strong approximation pairs can be found in Düntsch and Gediga [5].

6. The set $Form$ (the set of formulae) is given by the following inductive definition:

   (a) if $P \in \mathcal{P}(1)$ and $x \in Var$, then $P(x) \in Form$, and

   (b) if $A, B \in Form$, then $\neg A,\ (A \supset B),\ +A \in Form$.

7. The set Snt (the set of sentences) is given by the following definition: $Form \subset Snt$ and if $F \in Form$, then $\uparrow F \in Snt$, and $\downarrow F \in Snt$.

3.2. Interpretations

Let $L = \langle LC, Var, Con, \mathfrak{T}, Form, Snt\rangle$ be a language with approximative functors.

Definition 3.2. The ordered pair $\langle U, \varrho\rangle$ is an interpretation of L, if

1. U is a nonempty set;

2. ϱ is a function such that

   (a) $Dom(\varrho) = Con$

   (b) if $P \in Con$, then $\varrho(P) \in \{0, 1\}^U$;

Definition 3.3. Function v is an *assignment* relying on the interpretation $\langle U, \varrho\rangle$ if

1. $Dom(v) = Var$

2. $v(x) \in U$ for all $x \in Var$

3.2.1. Logically relevant general partial approximation space

Tools (the members of set $\mathfrak{T}$) determine a *logically relevant general partial approximation space* with respect to a given interpretation.

Definition 3.4. Let $Ip = \langle U, \varrho\rangle$ be an interpretation of L such that if $T \in \mathfrak{T}$, then $\varrho(T) \neq \emptyset$. The 5–tuple $\mathcal{PAS}(\mathfrak{T}) = \langle U, \mathfrak{B}, \mathfrak{D}_{\mathfrak{B}}, \mathsf{l}, \mathsf{u}\rangle$ is a logically relevant general partial approximation space with Pawlakian approximation pair generated by set $\mathfrak{T}$ of tools with respect to the interpretation, if $\mathfrak{B} = \{\varrho(T) : T \in \mathfrak{T}\}$.

3.2.2. Semantic Rules

1. If $T \in \mathfrak{T}$, then $[\![T]\!]_v = s$, where $s : U \to \{0, 1, 2\}$ is a function such that

$$s(u) = \begin{cases} 1 & \text{if } u \in \varrho(T) \\ 0 & \text{if } u \in \mathsf{l}(U \setminus \varrho(T)) \\ 2 & \text{otherwise} \end{cases}$$

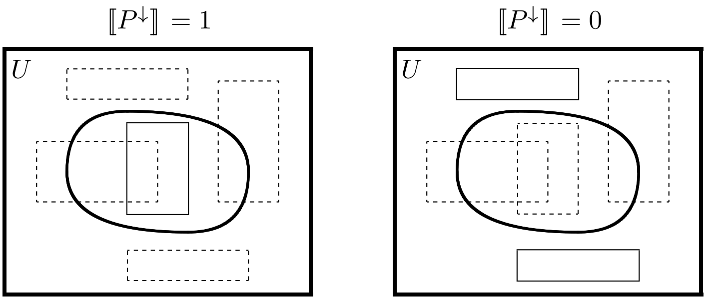

Figure 1: Illustration of lower approximation

2. If $P \in Pred$, then $[\![P^{\downarrow}]\!]_v = s$, where $s : U \to \{0, 1, 2\}$ is a function such that
$$s(u) = \begin{cases} 1 & \text{if } u \in \mathsf{l}([\![P]\!]_v) \\ 0 & \text{if } u \in \mathsf{l}(U \setminus \mathsf{u}([\![P]\!]_v)) \\ 2 & \text{otherwise} \end{cases}$$
Figure 1 shows the lower approximation of P. The oviform area in the middle represents the truth set of P, while the rectangles illustrate the truth sets of tools. Using the lower approximation, only the rectangle completely inside the oviform area belongs to the turth set of P.

3. If $P \in Pred$, then $[\![P^{\uparrow}]\!]_v = s$, where $s : U \to \{0, 1, 2\}$ is a function such that
$$s(u) = \begin{cases} 1 & \text{if } u \in \mathsf{u}([\![P]\!]_v) \\ 0 & \text{if } u \in \mathsf{l}(U \setminus \mathsf{u}([\![P]\!]_v)) \\ 2 & \text{otherwise} \end{cases}$$

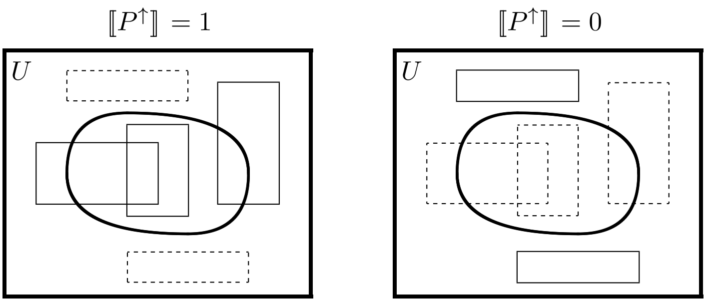

Figure 2: Illustration of upper approximation

Comparing figure 1 with 2, we see, that the approximate false set of P is the same. The truth set of P now is approximated with the tools represented by the union of the three big rectangles.

4. If $P \in \mathcal{P}(1) \setminus \mathfrak{T}$, then $[\![P]\!]_v = \varrho(P)$.
5. If $P \in \mathcal{P}(0)$ and $\circ$ is a sentence functor ($\circ \in \{\uparrow, \downarrow\}$) or missing, then $[\![\circ P(x)]\!]_v = [\![P^\circ]\!]\,(v(x))$
6. if $A, B \in Form$ and $\circ$ is a sentence functor ($\circ \in \{\uparrow, \downarrow\}$), or missing

$$[\![\circ{+}A]\!]_v = \begin{cases} 1 & \text{if } [\![\circ A]\!]_v = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$[\![\circ\neg A]\!]_v = \begin{cases} 2 & \text{if } [\![\circ A]\!]_v = 2 \\ 1 - [\![\circ A]\!]_v & \text{otherwise} \end{cases}$$

$$[\![\circ(A \supset B)]\!]_v = \begin{cases} 0 & \text{if } [\![\circ A]\!]_v = 1, \text{ and } [\![\circ B]\!]_v = 0; \\ 2 & \text{if } [\![\circ A]\!]_v = 2, \text{ or } [\![\circ B]\!]_v = 2; \\ 1, & \text{otherwise} \end{cases}$$

3.3. Central semantic notions

Let $L = \langle LC, Var, Con, \mathfrak{T}, Form, Snt\rangle$ be a given language with approximative functors, $\Gamma \subseteq Snt$ be a set of sentence and $A \in Form$ be a senence.

Definition 3.5. $\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a *possible representation* of L if

1. $\langle U, \varrho\rangle$ is an interpretation of L;
2. $\mathcal{PAS}(\mathfrak{T})$ is a logically relevant general partial approximation space with respect to the interpretation $\langle U, \varrho\rangle$;
3. v is an assignment relying on the interpretation $\langle U, \varrho\rangle$;

$\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a *representation* of Γ if

1. $\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a possible representation of L;
2. for all $A \in \Gamma$, $[\![A]\!]_v \neq 2$.

$\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a modell of Γ if

1. $\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a representation of Γ;
2. for all $A \in \Gamma$, $[\![A]\!]_v = 1$.

Let $\mathcal{PR}$ be a set of possible representations of language L.

1. Γ is satisfiable with respect to the set $\mathcal{PR}$ if it has a model in $\mathcal{PR}$.
2. A is a *weak semantic consequence* of Γ with respect to the set $\mathcal{PR}$ (in notation $\Gamma \vDash_w^{\mathcal{PR}} A$) if $\langle U, \varrho, \mathcal{PAS}(\mathfrak{T}), v\rangle$ is a $\mathcal{PR}$–modell of Γ, then $[\![A]\!]_v \neq 0$ (i.e., A is not false in any $\mathcal{PR}$–model of Γ).

3. A is *irrefutable* with respect to the set $\mathcal{PR}$ (in notation $\models_w^{\mathcal{PR}} A$) if $\emptyset \models_w^{\mathcal{PR}} A$ (i.e., A is never false in $\mathcal{PR}$).

Let $\mathcal{PR}$ be a set of possible representations of language L.

1. A is a *strong semantic consequence* of Γ with respect to the set $\mathcal{PR}$ (in notation $\Gamma \models_s^{\mathcal{PR}} A$) if

   (a) Γ has a representation in $\mathcal{PR}$;

   (b) every $\mathcal{PR}$–representation of Γ is a representation of $\{A\}$;

   (c) every $\mathcal{PR}$–model of Γ is a model of $\{A\}$.

2. A is *valid* with respect to the set $\mathcal{PR}$ (in notation $\models_s^{\mathcal{PR}} A$) if $\emptyset \models_s^{\mathcal{PR}} A$.

3.4. General and approximative laws

3.4.1. Law of non-contradiction

In classical case, the set $\{A, \neg A\}$ is unsatisfiable where $A \in Form$. The law it is still valid, using the lower or upper approximative sentence functors, so

- the set $\{\uparrow A, \uparrow \neg A\}$ is unsatisfiable, and
- the set $\{\downarrow A, \downarrow \neg A\}$ is unsatisfiable.

But in case when different sentence functors appears before formula A

- the set $\{\uparrow A, \downarrow \neg A\}$ is satisfiable, and
- the set $\{\downarrow A, \uparrow \neg A\}$ is satisfiable.

Theorem 3.6. *The sets $\{\uparrow A;\ \downarrow \neg A\}$, and $\{\downarrow A;\ \uparrow \neg A\}$ are satisfiable.*

Proof. Suppose, that we have a $P \in \mathcal{P}(1)$ one-argument predicate parameter. We are able to construct an interpretation $\langle U, \varrho\rangle$ and an assignment v such that there is an $u \in U$ such that $u \notin \mathsf{l}(\llbracket P \rrbracket)$, $u \in \mathsf{u}(\llbracket P \rrbracket)$ and $v(x) = u$. The evaluation results of the formulas $\uparrow P(x)$ and $\downarrow P(x)$ (with respect to $\langle U, \varrho\rangle$ and v) are different:

$$\llbracket \uparrow P(x) \rrbracket_v = 1, \quad \llbracket \downarrow P(x) \rrbracket_v = 2.$$

Let $A = +P(x)$, because $\llbracket \uparrow +P(x) \rrbracket = 1$ and $\llbracket \downarrow \neg + P(x) \rrbracket = 1$, therefore $\llbracket \uparrow A \rrbracket = 1$ and $\llbracket \downarrow \neg A \rrbracket = 1$ so $\{\uparrow A;\ \downarrow \neg A\}$ is satisfiable.
Let $A = \neg{+}P(x)$, because $\llbracket \uparrow \neg\neg + P(x) \rrbracket = 1$ and $\llbracket \downarrow \neg + P(x) \rrbracket = 1$, therefore $\llbracket \uparrow \neg A \rrbracket = 1$ and $\llbracket \downarrow A \rrbracket = 1$ so $\{\downarrow A;\ \uparrow \neg A\}$ is satisfiable. $\square$

3.4.2. Modus ponens

In the classical case $\{(A \supset B),\ A\} \vDash_w B$ and so the set $\{(A \supset B),\ A,\ \neg B\}$ is unsatisfiable, where $A, B \in Form$. The law it is still valid, using the lower or upper approximative sentence functors, so

- $\{\downarrow (A \supset B),\ \downarrow A\} \vDash_w \downarrow B$, and so the set $\{\downarrow (A \supset B),\ \downarrow A,\ \downarrow \neg B\}$ is unsatisfiable,
- $\{\uparrow (A \supset B),\ \uparrow A\} \vDash_w \uparrow B$, and so the set $\{\uparrow (A \supset B),\ \uparrow A,\ \uparrow \neg B\}$ is unsatisfiable.

But with different sentence functors (based on the idea described in Theorem 3.6)

- $\{\downarrow (A \supset B),\ \downarrow A\} \nvDash_w \uparrow B$ and so the set $\{\downarrow (A \supset B),\ \downarrow A,\ \uparrow \neg B\}$ is satisfiable,
- $\{\uparrow (A \supset B),\ \uparrow A\} \nvDash_w \downarrow B$ and so the set $\{\uparrow (A \supset B),\ \uparrow A,\ \downarrow \neg B\}$ is satisfiable.

3.4.3. Necessary conditions to use the approximation

After we showed the lack of connection between the formulae with different sentence functors, now we give a necessary condition to satisfy

$$[\![\Delta A]\!] = 1 \quad \Rightarrow \quad [\![\nabla A]\!] = 1$$

where A is an arbitrary formula and Δ, ∇ are different sentence functors, or one of them could missing.

- Let $P \in \mathcal{P}(1)$ and x is a variable. To satisfy the condition above, $[\![\Delta P(x)]\!] = 1 \ \Rightarrow \ [\![\nabla P(x)]\!] = 1$. Therefore, it is necessary for all $P \in \mathcal{P}(1)$:

$$[\![\Delta P(x)]\!] = 1 \quad \Rightarrow \quad [\![\nabla P(x)]\!] = 1$$

- When exists a substitution where $[\![\Delta P(x)]\!] = 0$, then $[\![\Delta\neg P(x)]\!] = 1$. To satisfy the condition above, $[\![\nabla\neg P(x)]\!] = 1$ must satisfy, so $[\![\nabla P(x)]\!] = 0$. Therefore it is necessary for all $P \in \mathcal{P}(1)$:

$$[\![\Delta P(x)]\!] = 0 \quad \Rightarrow \quad [\![\nabla P(x)]\!] = 0$$

The necessary conditions above are satisfied, where $\Delta = \downarrow$ and $\nabla = \uparrow$.

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