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Helical structure of space-filling mosaics based on 3D models of the 5D and 6D cubes

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Abstract

Our 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced either based on starting k edges arranged by rotational symmetry or as sequences of strut-chains originated from a separate one whose breakpoints join a single helix. Increasing the number of segments in the strut-chains to $k = n$ (infinity) creates continuous helices, whose Minkowski sum can be called n-zo-notope. By combining $2 < j < k$ edges, we can build 3-models of j-cubes, as parts of a k-cube. The suitable combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations keep the 3-model of the k-cube and necessary j-cubes derived from it and follow the helical structure of our models. Such a mosaic can have fractal structure as well, since we can replace it with a restructured one, built from multiplied solids. These are composed by the addition of 3-models of k- and j-cubes and are similar to the original ones. The intersections of a mosaic with planes allow limitless possibilities to produce periodical symmetric plane-tiling. Moving intersection planes results in a series of tessellations transforming into each other. Planar and spatial symmetry groups are the base of several works in different branches of art. Recent results can hopefully aid correlations between geometry, art and design.

Keywords: constructive geometry, hypercube modelling, tessellation, fractal, design

MSC: 52B10; 52B12, 52B15, 65D17

1. Three-models of k-cubes and tessellations

The discussed 3-dimensional framework (3-model) of any k-dimensional cube (k-cube) can be produced either based on starting k edges arranged by rotational symmetry or as sequences of strut-chains originated from a separate one whose breakpoints join a single helix. Increasing the number of segments in the strut-chains to $k = n$ (infinity) creates continuous helices, whose Minkowski sum can be called n-zonotope [1-3] (Figure 1). Combining $2 < j < k$ edges, we can build 3-models of j-cubes, as parts of a k-cube. Appropriate combinations of these zonotope models can result in 3-dimensional space-filling mosaics. The investigated periodical tessellations consist of the 3-model of the k-cube and necessary j-cubes derived from it [2-5] (Figures 2-3).

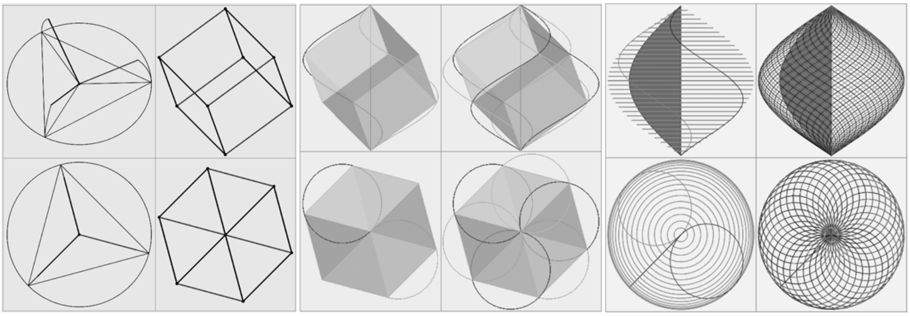

Figure 1: Three-dimensional modelling of higher-dimensional cubes

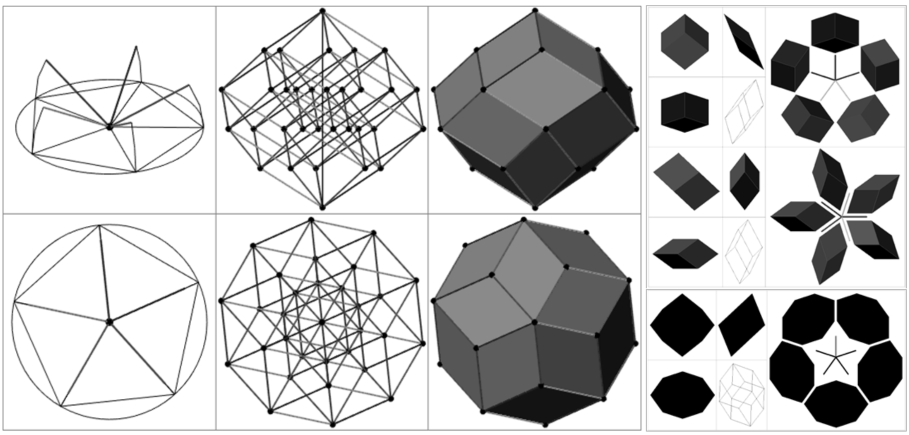

Figure 2: The 3-models of the 5-cube and of the derived j-cubes

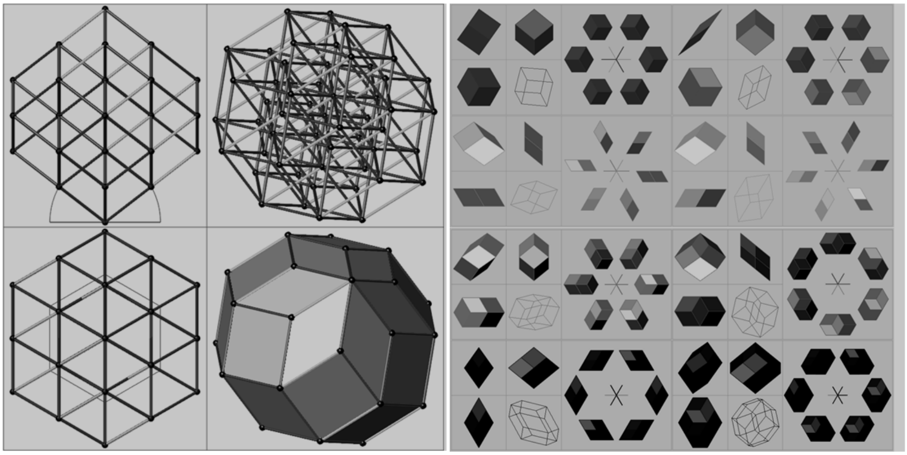

Figure 3: The 3-models of the 6-cube and of the derived j-cubes

2. Compound helical models and fractal tessellations

2.1. Six-dimensional cube

Figure 4 shows the densest example of the possible space-filling mosaics built from our 3-model of the 6-cube and derived parts of it. The proportion of the volume of the honeycomb of the periodical tessellation and the total volume of the applied 3-models of the 6-cube in it has the lowest value in this arrangement. The vertices of the stones join the breakpoints of chains of struts following symmetric pairs of left- and right-handed helices which are arranged in a well-known structure (Figure 5).

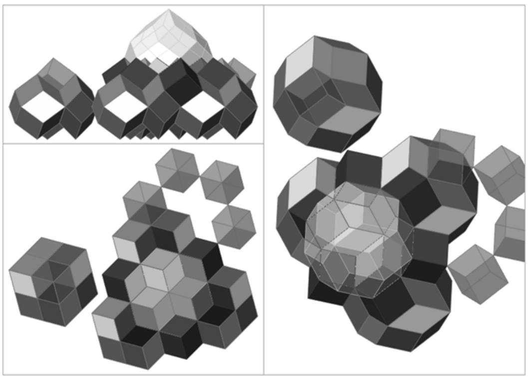

Figure 4: The initial space-filling periodical tessellation

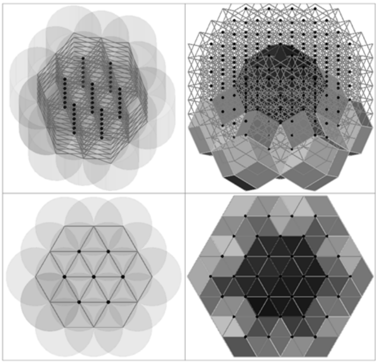

Figure 5: The helical layout of the vertices

The segmented helices, joining the outer and inner edges of the models, can be followed with the same set of stones having 1/6 edge-length (Figures 6-7). The compound 3-models of the derived 3- and 4-cubes are the common parts of this 6-cube model (Figure 7).

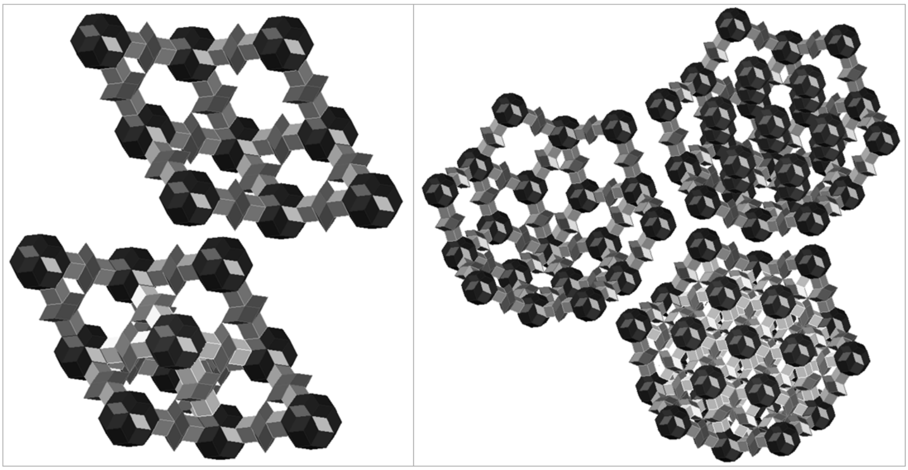

Figure 6: The inner structure of the compound 3-model of the derived 4-cube and the 6-cube

The remaining parts of the models can be filled out with the same set of stones again. If we rebuild the above space-filling mosaic with the new compound elements we will have the same arrangement of the same stones. The new structure can be distinguished if the parts have different colors according to their roles, such as elements of outer or inner edges, as well as filling stones. We took into consideration the number of dimension of the parts as well.

Figure 8 shows the alternating main sections (0, 1, 2, 3, 2, 1, 0, 1...) of our

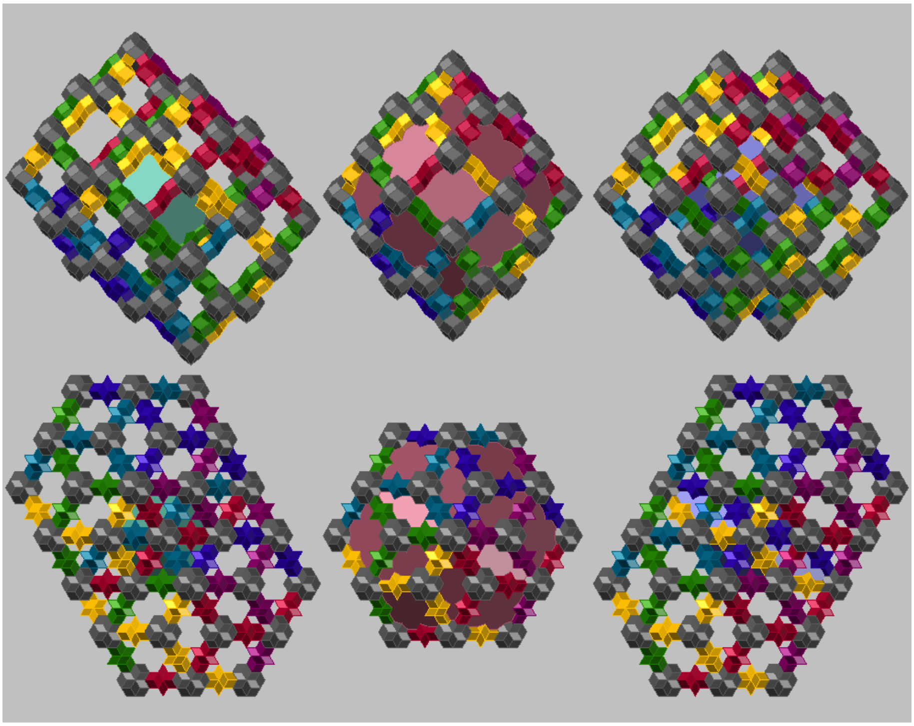

Figure 7: The compound 3-models of the derived 3- and 4-cubes are the common parts of this 6-cube model

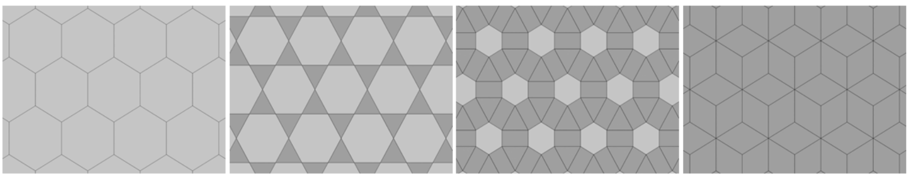

Figure 8: The alternating main horizontal sections of the tessellation showed in the figure 4

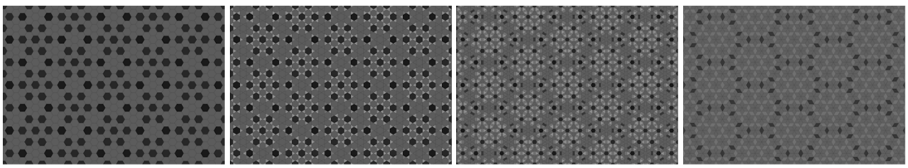

Figure 9: The patterns of the above sections after the first restructuring level

initial tessel-lation. These sections join the vertices of the stones. The geometrical patterns of the main sec-tions do not change even after infinitely repeated restructuring of the periodical space-filling mosaic. That means, a hidden fractal structure can be shown by the above coloring method. The sequence of the different sections, distinguished by colors, becomes more and more complex. Figure 9 shows the first segment of this sequence after the first restructuring level.

2.2. Five-dimensional cube

In the 5-dimensional case, the followed and the following cube models are affin pairs of each other (Figure 10). We do not construct the inner edges of the models this way but these can be filled with the same set of stones (Figure 11). We can build very dense tessellations with the resulting elements if we arrange these in honeycombs having parallelepiped and rhombic dodecahedron shapes. Two of the three results are new, compared to those described in [4-5]. The showed result is even built from a new set of elements (Figure 12).

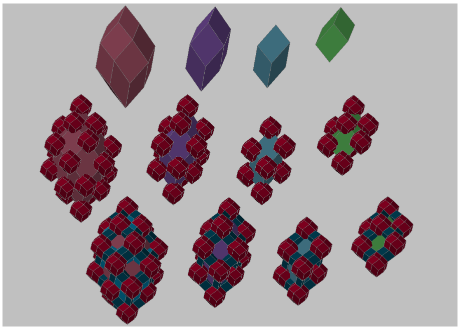

Figure 10: Compound 3-models in case of the 5-cube

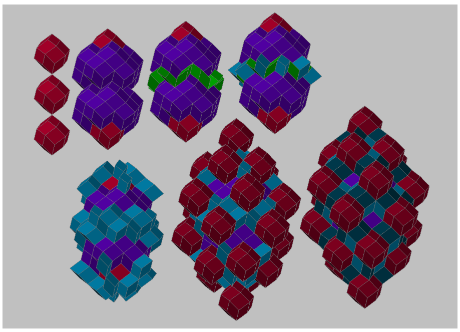

Figure 11: Filling of the compound model of the 5-cube

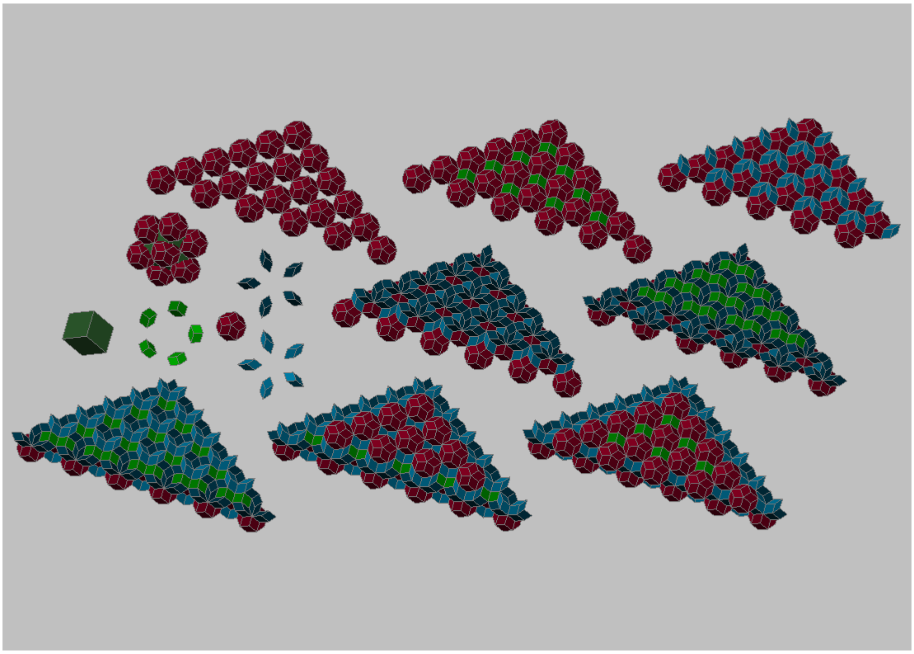

Figure 12: New tessellation gained from an earlier one by restructuring with the new compound models

The stones of the mosaics can be replaced with the compound models and this process can be repeated infinitely. We may have a fractal like structure of the spatial tessellations.

A more sophisticated compound model consists of the outer and inner edges as well. These are followed with our 3-model of the 5-cube and the derived 4-dimensional element (Figure 13). The inner edges intersect each other according to the golden mean. The remaining parts of the model can be filled with the same elements and 3-models of the derived 3-cubes. The whole filled compound model consists of 4588 stones.

The inner structure could be illustrated by a vertically moved horizontal intersecting plane. Figure 14 shows all second, differently patterned and colored, main horizontal sections of our compound model. The parts are colored according to their roles like elements of outer or inner edges as well as filling stones. We also took into consideration the number of dimensions of the parts as well. The appropriate parts of our model can constitute the derived lower-dimensional elements. We could replace the stones of the known tessellations with the new compound models and repeat this replacement infinitely. That means, we could gain new fractal like periodical space-filling mosaics. To accomplish this would require a powerful computer.

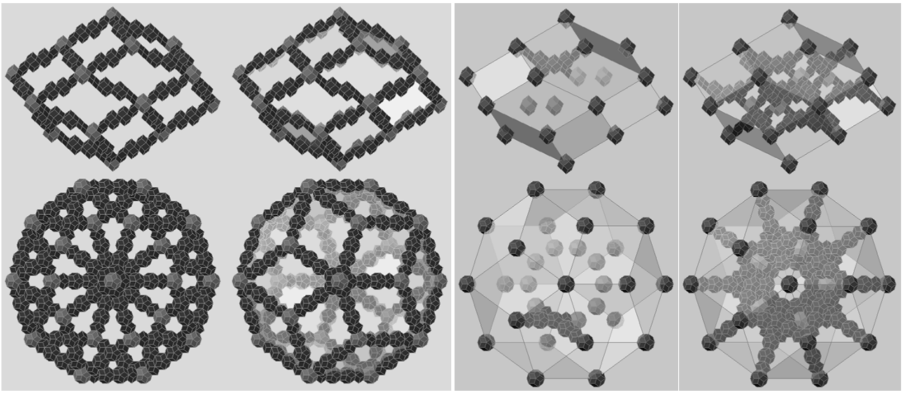

Figure 13: More sophisticated compound 3-model of the 5-cube consisting of the outer and inner edges

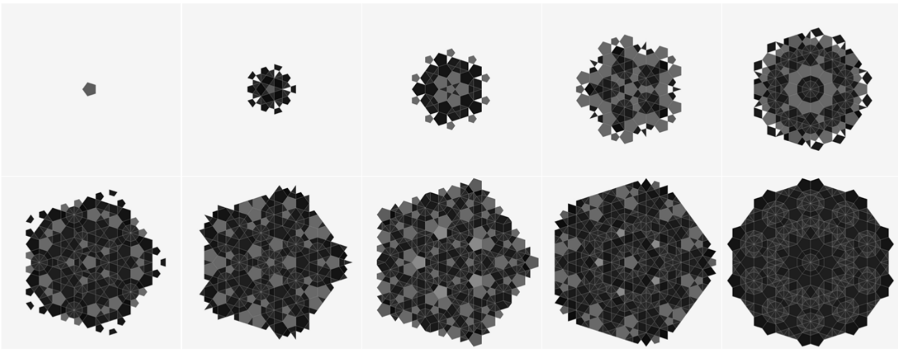

Figure 14: All second, differently patterned and coloured, main horizontal sections of the above model

3. Concluding remarks

3.1. Connections to arts

The intersections of the mosaics with planes allow unlimited possibilities to produce perio-dical symmetric plane-tiling. Moving intersection planes result in series of tessellations trans-forming into each other which can be shown using various animations. These provide possibilities for exhibitions, publicity work and further usage. Planar and spatial symmetry groups are the base of several works in different branches of art. Our symmetric models of the hypercube and the symmetrically arranged periodical tessellations offer several binding points to this field [3-5]. Recent results can hopefully foster newer connections between geometry, art and design.

3.2. Technical remarks

The creation of the constructions and figures for this paper was aided by the AutoCAD program and AutoLisp routines developed by the author. The solid modeling and visualizing of the results have their own limits corresponding to software and hardware. Extending this topic to higher dimensional cubes would also require a powerful computer.

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