Bivariate difference-differential dimension polynomials and their computation in MAPLE™

Christian Dönch[†], Franz Winkler[†]
RISC, JKU, Linz, Austria
[cdoench@risc.jku.uni-linz,.ac.at, winkler@risc.uni-linz.ac.at]
2000 Mathematics Subject Classification. 12H99

Let K be a field and let $\{d_1, \ldots, d_m\}$ and $\{s_1, \ldots, s_n\}$ be two sets of mutually commuting derivatives and automorphisms acting on K, respectively. Let $\Lambda := \{\delta^k\sigma^l \mid \delta = (\delta_1, \ldots, \delta_m),\ \sigma = (\sigma_1, \ldots, \sigma_n),\ k \in \mathbb{N}^m,\ l \in \mathbb{Z}^n\}$ denote the corresponding set of difference-differential terms. We use the notation $(l_1, \ldots, l_n)_{|\cdot|}$ to denote the multi-index $(|l_1|, \ldots, |l_n|)$. Then the differential- and difference order of $\lambda = \delta^k\sigma^l$ are given by $\mathrm{ord}_\delta(\lambda) := |k|$ and $\mathrm{ord}_\sigma(\lambda) := |l_{|\cdot|}|$, respectively. By D we denote the free K-module generated by Λ. For $\theta = \sum_{\lambda\in\Lambda} a_\lambda \lambda \in D$ let the differential- and difference-order of θ be given by $\mathrm{ord}_\delta(\theta) := \max_{a_\lambda \neq 0}\{\mathrm{ord}_\delta(\lambda)\}$ and $\mathrm{ord}_\sigma(\theta) := \max_{a_\lambda \neq 0}\{\mathrm{ord}_\sigma(\lambda)\}$, respectively. By D_{rs} we denote the subset of elements of D whose differential- and difference-order are bounded by r and s, respectively. D can be equipped with a natural ring structure by the commutation rules $\alpha\beta = \beta\alpha$, $\delta_i a = a\delta_i + d_i(a)$, $\sigma_j a = s_j(a)\sigma_j$ for all $a \in K$, $\alpha, \beta \in \{\delta_1, \ldots, \delta_m, \sigma_1, \ldots, \sigma_n\}$, $1 \leq i \leq m$, $1 \leq j \leq n$. For a left D-module M generated by $h_1, \ldots, h_q$ let $M_{rs} := D_{rs}h_1 + \ldots + D_{rs}h_q$. Then there exists a polynomial $\psi(r, s) \in \mathbb{Q}[r, s]$ such that for all $r, s \in \mathbb{N}$ sufficiently large $\psi(r, s) = \dim_K M_{rs}$. The polynomial ψ is called the *bivariate difference-differential dimension polynomial* of M. This concept is similar to the one of Hilbert-polynomials in commutative algebra. A. Levin gave a characteristic set approach for computing ψ [Lev00], and a very general Gröbner bases approach for computing multivariate dimension polynomials [Lev07], where the Gröbner bases are computed with respect to several orderings but the automorphisms are not considered to be invertible. M. Zhou and F. Winkler published an algorithm for computing Gröbner bases with respect to two orderings for the case of invertible automorphisms and used this to compute ψ in a fashion similar to Levin's [ZW08]. We are going to recall the basic notions of their approach in order to present MAPLE™ implementations of algorithms for computing Gröbner bases in difference-differential modules and for computing bivariate difference-differential dimension polynomials. A detailed description of the implementations can be found in [Dön09].

[Dön09] C. Dönch, *Bivariate difference-differential dimension polynomials and their computation in* MAPLE™ , to appear

[Lev00] A. Levin, *Reduced Gröbner Bases, Free Difference-Differential Modules and Difference-Differential Dimension Polynomials*, JSC 30 (2000), 357-382

[Lev07] A. Levin, *Gröbner bases with respect to several orderings and multivariable dimension polynomials*, JSC 42 (2007), 561-578

[ZW08] M. Zhou, F. Winkler, *Computing difference-differential dimension polynomials by relative Gröbner bases in difference-differential modules*, JSC 43 (2008), 726-745

[†]This work has been supported by the Austrian FWF project P20336-N18